

Duality and topology

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Mappings between models may be obtained by unitary transformations with preservation of the spectra but in general a change in the states. Non-canonical transformations in general also change the statistics of the operators involved. In these cases one may expect a change of topological properties as a consequence of the mapping. Here we revisit some dualities resulting from the mappings and introduce new ones, by systematically using a Majorana fermion representation of spin and fermionic problems. We focus on the change of topological invariants that results from unitary transformations, either non-canonical with change of statistics or canonical transformations, taking as examples the mapping between a spin system and a topological superconductor and between different fermionic systems. As a biproduct one may rewrite strongly interacting systems as weakly interacting Hamiltonians, specifically the Hubbard model.

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I. INTRODUCTION

Many-body interacting systems are problems that are hard to solve, often requiring non-perturbative approaches to properly describe their cooperative phenomena. In general the complexity of the problem may be reduced identifying the dominant modes that govern the behavior of the system, particularly when a low-energy description is enough. As a consequence, an important simplification results if one is able to identify appropriate variables or operators associated with those dominant modes.

Many authors have considered various transformations between variables or operators (depending if the system is classical or quantum, respectively) to obtain a good description of the system behavior and, in a few cases, solve exactly the problem. Typically a goal has been to simplify strongly correlated systems by reducing the strongly interacting terms to free terms, with the usual price of rendering simpler original free terms into complicated weakly interacting terms. The procedure is argued to be worthwhile when the strength of the interacting term is large with respect to the non-interacting term. This may allow a reasonable description of the system using perturbation theory in the original free term or using some sort of mean-field decoupling of the generated complex term.

Depending on the problem, transformations from one set of operators to another may involve canonical transformations¹, preserving the statistics, or non-canonical transformations, where the statistics is altered. Typical examples are bosonization of a fermionic problem or, reversely, fermionization. Also, often it is convenient to transform between spin problems and fermionic problems. Some transformations are exact²⁻⁴ but in some cases there is an enlargement of the Hilbert space, and a projection to the physical subspace is required⁵⁻¹¹. In general this projection can only be implemented on

average. The simplest cases involve local transformations from one set of operators to another, but non-local transformations are also convenient in some cases. Also, some transformations have an intrinsic non-linear character¹. A familiar example are the bilinear representations of spin operators in terms of bosonic or fermionic operators¹².

In this work we will focus on fermionic and spin-1/2 systems. Among the many representations for spin systems one is particularly convenient since it allows an exact preservation of the commutation relations of the spin operators¹³⁻¹⁵. There is some enlargement of the Hilbert space but it just leads to a multiplying factor in the partition function of the system¹⁶⁻¹⁸. Specifically, the spin operators may be represented by a bilinear representation in terms of three Majorana operators. Majorana operators may also be used to represent a fermionic operator in a simple way. A fermionic operator may be understood as containing a real and imaginary parts if these are chosen as hermitian operators, which is the characteristic property of a Majorana fermion. Clearly then we may use Majoranas to represent both spin-1/2 operators and usual fermionic operators and therefore may look for relations between them. It is therefore convenient to look for transformations between spin and fermionic operators using a language of transformation operators in terms of Majorana fermions.

Since the transformation between the two sets of operators is bilinear, it naturally provides a way for a non-canonical transformation with consequent change of statistics. However, looking for general transformations using Majorana operators, it is possible to consider different choices of representations which enable both canonical and non-canonical transformations, as we will expand later on. In addition, it has been proposed that they provide a convenient way to understand the general properties of transformations between operators^{19,20}.

The spin-1/2 representation in terms of Majoranas has

been applied in several contexts from problems in the context of the Kondo problem^{14,17}, to the Ising model in a transverse field¹⁵ and the Heisenberg model¹⁸. Also, non-equilibrium problems have been addressed using this method^{21,22}.

The interest in Majorana fermions has recently been revived due to their possible relevance in quantum computation problems, due to their non-abelian character, and have been proposed to be observed in the context of topological superconductors²³. Even though appropriate materials are difficult to find in nature, several engineered possibilities have been proposed such as semiconductor wires with strong spin-orbit coupling placed on top of a conventional superconductor and in the presence of a Zeeman field²⁴, or magnetic impurities on top of a conventional superconductor²⁵. In these systems the Majorana modes are associated with edge modes at the border between a topologically non-trivial system and a trivial system (such as the vacuum outside the material).

The transformation between sets of operators or variables leads to an equivalent problem whose Hamiltonian is expressed in terms of other physical quantities. In some loose sense we may think of a transformation as a relation between dual Hamiltonians. Dualities appear in physics in various contexts from dualities in classical electromagnetism (between electric and magnetic fields at the level of Maxwell's equations), dualities in statistical physics problems (such as the Kramers-Wannier identities²⁶ that relate the partition function of the system in the low and high temperature limits) and dual lattices (by establishing a relation between variables or operators defined at the corners or links of some lattice problem). For instance in the last example in the context of spin problems the role of the interaction term and a magnetic field term are reversed as one goes from a site description to a bond-variable description.

As pointed out recently²⁷ the duality transformations do not need to relate strong and weak coupling regimes (although they are particularly useful when this occurs), and usually are non-local relationships involving often some kind of extended strings of operators. Intrinsic to the idea of duality and the search of an alternative set of operators to describe the properties of some Hamiltonian, is the equivalence between the two descriptions. An expected minimal requirement is that the spectrum of the Hamiltonian is preserved. An obvious way to achieve this is to consider unitary transformations since the spectrum is preserved. However, in general, the states are not preserved even though the relation between them is determined by the choice of the unitary operator. As a consequence of the change of states it has been pointed out that, in general, level degeneracy changes and therefore the correspondence between the two Hamiltonians is not complete. In this work we will however explore the effect of unitary operators.

It has also been pointed out recently that the usual non-local character of the duality transformation suggests that the use of bond operators may be more conve-

nient than local relations. In reference²⁷ several transformations have been considered and the standard example of a Jordan-Wigner transformation between a spin-1/2 problem and a spinless fermion non-interacting Hamiltonian has been derived with the bond-duality approach. In the case of nearest-neighbor couplings this transformation has been known for a long time to fully diagonalize the Hamiltonian, since in the fermionic language is non-interacting. However, it is also known that the Jordan-Wigner transformation may be seen as the result of a local unitary transformation¹⁷ (understood as a product of local transformations across all lattice sites). In this work we will follow a similar procedure and therefore consider a less stringent definition of duality (see also, for instance²⁸).

The mapping between the XY spin model in a transverse field and the Kitaev superconducting model²⁹ reveals another interesting result. While the original spin problem is topologically trivial, the resulting transformed Hamiltonian has topological regimes. Therefore, as a consequence of the exact transformation between the two problems, it appears that the topological properties have changed.

Topological systems appear in various contexts. To name a few, among those that have received recently considerable attention are the topological insulators and topological superconductors^{30,31} due to their robust edge states with possible applications in dissipation free transport and quantum computation, as mentioned above. Other well-known topological systems are some spin chains.

An example of phases with topological origin are the gapped Haldane phases³² of odd integer spin chains. These are distinguished from the half-integer gapless phases and the gapped even integer spin chains that are topologically trivial. In general topological order is not associated with local order parameters and therefore do not fall in the category of symmetry broken phases in the usual meaning of the Landau theory. However, there may be some coexistence of topological properties and some form of a local order parameter, which is however unrelated with the topological properties. An example is some symmetry-protected phases of some topological systems such as a topological superconductor where the pairing order parameter has some fixed phase. In generality superconductors are topologically ordered³³, however.

The Haldane phase conjectured in integer spin chains finds an exact realization in the gapped AKLT phases³⁴ where non-local string order has been found. This hidden order is the result of a hidden order symmetry and as consequence one may introduce a string order parameter^{35,36}. Associated with the topological phase are free modes at the edges of the chain that are protected by one of various symmetries. This hidden order and associated symmetry has been understood as the result of a non-local unitary transformation^{37,38} and it has been argued that its non-local character may imply long-range order in the transformed Hamiltonian. It was

also argued that symmetry breaking in the transformed Hamiltonian is an indication of edge states in the original Hamiltonian opening a way to explore possible order parameters in topological systems related to other models with clear order parameters in the Landau sense. Therefore one may search for correlation functions in a topological system that are related via duality with the order parameters in the trivial but ordered dual system^{39,40}.

We note at this point an important difference between systems with topological order and systems that are not topologically ordered but have topological properties, characterized by some topological invariants and due to the bulk-edge correspondence edge modes protected by some symmetries. This last group of topological systems only display short-range topological order that by definition may be washed away by some appropriate local transformations. True topologically ordered systems display long-range topological order that may be not eliminated by any local transformation^{41,42}.

While there are different definitions of topological order in the literature^{41,43} it will not be relevant in this work since we will focus on the simpler class of symmetry-protected topological systems. Also, for simplicity we will focus on one-dimensional systems and on the mapping between spin systems and fermionic systems. We will focus on the use of unitary transformations applied to the Hamiltonian to transform the description of the problem using a different set of operators. This will be implemented within a Majorana fermion description that unifies the two sets of problems and the transformations can be understood as some reduction or enlargement of the degrees of freedom. The mapping also allows a new formulation for the Hubbard model which can be extended to other strongly interacting problems.

In some cases it has been shown that is possible to reduce an apparent interacting term into a free problem, with the consequent exact solution. An example is provided by a X-Y chain which may be reduced to a problem of free (in the sense of quadratic) problem of spinless fermions. This example illustrates a problem of statistical transmutation of an original problem of spins into a problem of fermions. This is well known to be achieved applying a Jordan-Wigner transformation which traditionally is understood as a transformation between operators defined such that their commutation relations are satisfied. Other non-canonical transformations include the Schrieffer-Wolff transformation⁴⁴ between the Anderson model and the Kondo model and bosonization between fermions and a bosonic field⁴⁵.

This work has a few goals:

- i) We use local unitary transformations to describe strongly interacting systems by reducing their interacting terms into free terms. The results are applied to the Hubbard model and a new description is obtained.
- ii) Even though this type of problems has been considered by many authors before, other approaches use mainly transformations between the operators and new types of operators believed to provide a better and sim-

pler description of the physics of the problem. Here we attempt to achieve the same goal with no direct transformations between operators but by transforming directly the Hamiltonian, described in terms of Majorana operators. The procedure implies applying a unitary transformation such that interacting terms (quartic in the Majoranas) are replaced by quadratic terms in terms of the same or others Majoranas. The two procedures should be equivalent.

iii) As mentioned above, a similar procedure has been followed before to reduce the X-Y chain to a problem of free spinless fermions¹⁷. In that specific problem the transformation used in addition to be unitary is also hermitian. This leads to the condition that $U^2 = I$, where U is the transformation and I the identity. Therefore, we consider a set of families of operators that obey this restriction and apply them to spin problems.

iv) Majoranas allow a representation of both spin problems and fermionic problems. Starting from a spin problem ($S = 1/2$) we need three Majorana operators to represent them. Spinless fermions may be represented by two Majoranas. At most we need therefore three Majoranas, and we have considered a general transformation, U , that involves the possible powers of three Majorana operators. Therefore, a general transformation may be written as $U = zI + wI_3 + \vec{x} \cdot \vec{\gamma} + \vec{y} \cdot \vec{S}$, where I is the identity, γ_i are the three components of the Majorana operators, $I_3 = -i\gamma_1\gamma_2\gamma_3$ (which is also a Majorana operator) (we use the normalization $\gamma_i^2 = 1$) and the spin operator is defined by $S_x = -(i/2)\gamma_2\gamma_3$, $S_y = -(i/2)\gamma_3\gamma_1$, $S_z = -(i/2)\gamma_1\gamma_2$. Actually it is convenient to define a related "spin" operator defined by $\vec{\mathcal{S}} = 2\vec{S}$. We consider both transformations which are unitary and unitary and hermitian.

v) As in the transformation between the spin X-Y model and the Kitaev model it is possible to transform between a system that has Landau-like transitions between disordered and ordered (long-range) phases to a system that has transitions between trivial and topological phases. It is one goal to understand how we may transform between a system with no topological properties and a system with topological properties. In this case this occurs in a problem with statistical transmutation. In addition to the change of the representation of the Hamiltonian in terms of new operators, we analyse the change of the representation of the states of a given fermionic Hamiltonian.

vi) It has also been shown recently by other authors that it is possible to transform topological insulators to topological superconductors⁴⁶. In this case this is a canonical transformation. It was shown that topological phases are matched to topological phases and trivial phases to trivial phases. This suggests that a non-canonical transformation is required to change topology. As we will show later strings appear in this problem since one applies a local transformation at all lattice sites. When we reduce a spin operator (two Majoranas) to a Majorana operator (one Majorana) the parity of the op-

erators changes. This implies that the unitary operator applied at other site locations does not commute any more with a given (different) lattice site where the transformation took place, if in the unitary operator there are odd powers of Majorana operators: this gives rise to extra terms appearing in the Hamiltonian that are the so-called strings. Of course like in the case of the Jordan-Wigner transformation, these strings are not apparent in the Hamiltonian if the interactions between the spins are between nearest neighbors. However, they do appear if the hoppings have longer range.

vii) Considering spinfull problems, such as the Hubbard model or the Kondo model, one needs four Majorana operators to represent the fermionic operators. The general expansion of the unitary operator now involves more terms going up to a product of four Majorana operators and including all possible powers of three Majorana operators. In the case of the Kondo model it has been proposed^{47,48} that we may substitute one of the four operators, say γ_4 , by the product of the other three such as $\gamma_4 = i\gamma_1\gamma_2\gamma_3$. So we are back to a description similar to spin problems. In the case of the Hubbard model the U term involves all four Majorana operators. It turns out that this term is invariant to any unitary operation. Other ways, such as operator transformations, are available and allow a reduction of the interacting term to a free term^{49,50}. But here we do not want to follow that route. As shown before⁵¹ we may enlarge the space (to other ghost four Majorana operators) and proceed with similar schemes. Here we also enlarge the space but by going from 4 Majorana operators to 6 Majorana operators (instead of 8 operators). We represent the interacting term in terms of fictitious spin operators (not related to the spin of the electrons) and show that it is enough to consider two spin 1/2 operators (and therefore in terms of Majoranas two sets of three operators). In this enlarged space we are able to reduce the interacting term to a free term. Of course, as in other approaches the hopping becomes now quite complex and probably some mean-field approach is required.

viii) The main focus of the work is however the question of apparent topology change as we transform the representation of the Hamiltonian. Taking the example of the mapping between the XY spin model and a Kitaev-like fermionic model the original model can be seen as an interacting system in terms of the Majorana operators. The calculation of a topological invariant may be carried out using a Green's function approach⁵² or using twisted boundary conditions⁵³, for instance. Such treatments require a numerical solution. The mapped problem is however easily diagonalizable and its topological properties easily obtained. However, the choice of operator representation leads to an apparent change of topology that is considered further ahead.

In this work we follow a pedestrian approach to the problem introducing new results and reformulating results obtained previously, using systematically a representation of the unitary operators in terms of Majorana

fermions. This allows in a natural way the relation between spin systems and fermionic systems. In section II we review the Jordan-Wigner transformation that allows the mapping between a nearest-neighbor spin problem and a system of free spinless fermions. In section III we briefly recall the Majorana representation of the Kitaev model for a one-dimensional triplet superconductor. In section IV we consider the non-canonical unitary (and hermitian) transformation that the Jordan-Wigner transformation implements. In section V we generalize the procedure to other unitary transformations. In section VI we propose a description of the Hubbard model that may be suitable to study the strong coupling limit. In section VII we establish the relation between the states in the two descriptions of the problem of the XY spin chain. In section VIII we establish the relations between the topological invariants between two systems that are related by a unitary transformation. Also, the non-local transformation that is the result of the diagonalization of the Kitaev model at some special points is considered and the change of topological numbers is related to the singular character of the transformation. We conclude with section IX.

II. JORDAN-WIGNER TRANSFORMATION

We begin by considering the standard problem of a spin-1/2 chain, where the spin operators lie in the XY plane isotropically. The Hamiltonian is described by

$$H = \frac{J}{2} \sum_{j=1}^N (S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-) \quad (1)$$

This is an interacting quantum problem that is well known to be diagonalizable. A possible way consists in performing the transformation from the spin operators to spinless fermionic operators as⁵⁴

$$\begin{aligned} S_j^+ &= c_j^\dagger e^{i\pi \sum_{n=1}^{j-1} c_n^\dagger c_n} \\ S_j^- &= e^{-i\pi \sum_{n=1}^{j-1} c_n^\dagger c_n} c_j \end{aligned} \quad (2)$$

As a result the Hamiltonian transforms to a free fermionic expression

$$H = \frac{J}{2} \sum_{j=1}^N (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \quad (3)$$

Using periodic boundary conditions, $S_{N+1}^\alpha = S_1^\alpha$, with $\alpha = x, y, z$, we get that

$$\begin{aligned} c_{N+1} &= e^{i\pi \sum_{n=1}^N S_n^+ S_n^-} S_{N+1}^- \\ &= e^{i\pi \sum_{n=1}^N (\frac{1}{2} + S_n^z)} S_1^- \end{aligned} \quad (4)$$

Using that $C_1 = S_1^-$ we obtain that $c_{N+1} = i^N e^{i\pi S^z}$, $c_1 = e^{i\pi N_F} c_1$. Here

$$S^z = \sum_{n=1}^N S_n^z = \sum_{n=1}^N \left(c_n^\dagger c_n - \frac{1}{2} \right) = N_F - \frac{N}{2} \quad (5)$$

where N_F is the number of fermions. If a state has $S^z = 0$ then the number of fermions is $N_F = N/2$ and in general the number of fermions is $N_F = S^z + N/2$.

We may consider a system that has anisotropy in the XY plane described by the Hamiltonian

$$H_{XY} = - \sum_{j=1}^N (J_x S_j^x S_{j+1}^x + J_y S_j^y S_{j+1}^y) \quad (6)$$

Defining as above $S^x = (S^+ + S^-)/2$; $S^y = (S^+ - S^-)/(2i)$ we can write that

$$H_{XY} = -\frac{1}{4} \sum_j [(J_x - J_y) (S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-)] \\ -\frac{1}{4} \sum_j [(J_x + J_y) (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+)] \quad (7)$$

Performing a Jordan-Wigner transformation we obtain that

$$H_{XY} = -\frac{1}{4} \sum_j [(J_x - J_y) (c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j)] \\ -\frac{1}{4} \sum_j [(J_x + J_y) (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)] \quad (8)$$

This model is related to the Kitaev model (at vanishing chemical potential) if we rewrite it as

$$H = -t \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta \sum_j (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \quad (9)$$

choosing $t = (J_x + J_y)/4$; $\Delta = (J_x - J_y)/4$. Therefore if $J_y = 0$ we get that $\Delta = t$ and if $J_x = J_y$ we get that $\Delta = 0$.

III. MAJORANA REPRESENTATION OF THE KITAEV MODEL

Consider now the Kitaev model, but with the addition of a chemical potential term, $\mu \neq 0$. This can be traced back to a magnetic field in the XY model. The Hamiltonian is written as

$$H = -t \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu \sum_j \left(c_j^\dagger c_j - \frac{1}{2} \right) \\ + \Delta \sum_j (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \quad (10)$$

The spinless fermionic operators may be written in terms of Hermitian operators, Majorana operators, as

$$c_j = \frac{1}{2} (\gamma_{1,j} + i\gamma_{2,j}) \\ c_j^\dagger = \frac{1}{2} (\gamma_{1,j} - i\gamma_{2,j}) \quad (11)$$

In terms of these operators the Hamiltonian can be written as

$$H = \frac{i}{2} \sum_j [(-t + \Delta) \gamma_{1,j} \gamma_{2,j+1} + (t + \Delta) \gamma_{2,j} \gamma_{1,j+1}] \\ - \frac{i}{2} \mu \sum_j \gamma_{1,j} \gamma_{2,j} \quad (12)$$

Note that at some particular point ($\mu = 0, t = \Delta$) the Hamiltonian becomes quite simple

$$H = it \sum_j \gamma_{2,j} \gamma_{1,j+1} \quad (13)$$

IV. NON-CANONICAL TRANSFORMATION: X-Y MODEL TO KITAEV MODEL

Let us start from the spin-1/2 XX model

$$H_{XX} = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) \\ = \frac{1}{2} \sum_j (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \quad (14)$$

Also, we will consider the fully anisotropic X model

$$H_X = \sum_j (S_j^x S_{j+1}^x) \\ = \frac{1}{4} \sum_j (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-) \quad (15)$$

Let us represent the spin operators by Majorana operators as^{13,14}

$$S^x = -\frac{i}{2} \gamma_2 \gamma_3 \\ S^y = -\frac{i}{2} \gamma_3 \gamma_1 \\ S^z = -\frac{i}{2} \gamma_1 \gamma_2 \quad (16)$$

and, therefore, $S^+ = \gamma_3 (\gamma_1 + i\gamma_2)/2$, $S^- = (\gamma_1 - i\gamma_2) \gamma_3/2$. Define now usual fermionic operators (non-hermitian) as

$$g = \frac{1}{2} (\gamma_1 - i\gamma_2) \\ g^\dagger = \frac{1}{2} (\gamma_1 + i\gamma_2) \quad (17)$$

We get that $\gamma_1 = g + g^\dagger$ and $i\gamma_2 = g^\dagger - g$. Therefore we may write the Hamiltonians as

$$H_{XX} = \frac{1}{4} \sum_j (\gamma_{1,j} \gamma_{1,j+1} + \gamma_{2,j} \gamma_{2,j+1}) \gamma_{3,j} \gamma_{3,j+1} \\ H_X = \frac{1}{4} \sum_j \gamma_{2,j} \gamma_{2,j+1} \gamma_{3,j} \gamma_{3,j+1} \quad (18)$$

or

$$\begin{aligned}
H_{XX} &= \frac{1}{2} \sum_j \left(g_j^\dagger g_{j+1} + g_j g_{j+1}^\dagger \right) \gamma_{3,j} \gamma_{3,j+1} \\
H_X &= \frac{1}{4} \sum_j \left(g_j^\dagger g_{j+1} + g_j g_{j+1}^\dagger - g_j^\dagger g_{j+1}^\dagger - g_j g_{j+1} \right) \gamma_{3,j} \gamma_{3,j+1}
\end{aligned} \quad (19)$$

Confirming the interacting nature of the spin problem, the Hamiltonian in terms of the Majorana and regular fermions is also interacting, as evidenced by the quartic terms in the Hamiltonian. A possible way to diagonalize the Hamiltonian is to perform a unitary transformation that eliminates the γ_3 operators. This can be achieved using a local operator¹⁷

$$U_{z,j} = \left(1 - g_j^\dagger g_j \right) + \gamma_{3,j} g_j^\dagger g_j \quad (20)$$

In addition to be a unitary operator it turns out that this operator is also hermitian. Therefore $U_{z,j} = U_{z,j}^\dagger$; $U^2 = I$. It can be easily shown that

$$\begin{aligned}
U_{z,j} \gamma_{3,j} g_j^\dagger U_{z,j}^\dagger &= g_j^\dagger \\
U_{z,j} g_j \gamma_{3,j} U_{z,j}^\dagger &= g_j
\end{aligned} \quad (21)$$

Define now an operator that is the product over all sites in the one-dimensional system

$$U_z = \prod_{j=1}^N U_{z,j} = U_{z,1} \cdots U_{z,N} \quad (22)$$

Consider now the action of this operator on a local operator at site j :

$$U_z \left(\gamma_{3,j} g_j^\dagger \right) U_z^\dagger = \left(\prod_{l=1}^{j-1} U_{z,l} \right) g_j^\dagger \left(\prod_{l=j-1}^1 U_{z,l}^\dagger \right) \quad (23)$$

Note now that

$$U_{z,l} g_j^\dagger U_{z,l}^\dagger = g_j^\dagger \left(1 - 2g_l^\dagger g_l \right) = g_j^\dagger (-1)^{n_l} \quad (24)$$

So,

$$\begin{aligned}
U_z \gamma_{3,j} g_j^\dagger U_z^\dagger &= (-1)^{\sum_{l=1}^{j-1} n_l} g_j^\dagger \\
U_z g_j \gamma_{3,j} U_z^\dagger &= (-1)^{\sum_{l=1}^{j-1} n_l} g_j
\end{aligned} \quad (25)$$

We see that this unitary transformation gives origin to the so-called strings associated with the occupation of states to the left of a given lattice site, j . This is like in the Jordan-Wigner transformation. Actually this unitary transformation leads exactly to the Jordan-Wigner transformation if applied to the Hamiltonian¹⁷. We will show this in the following.

Let us act with the operator U_z on the spin Hamiltonians:

$$\begin{aligned}
U_z H_{XX} U_z^\dagger &= \frac{1}{2} \sum_j \left[U_z g_j^\dagger \gamma_{3,j} U_z^\dagger U_z \gamma_{3,j+1} g_{j+1} U_z^\dagger \right. \\
&\quad \left. + U_z \gamma_{3,j} g_j U_z^\dagger U_z g_{j+1}^\dagger \gamma_{3,j+1} U_z^\dagger \right] \\
&= -\frac{1}{2} \sum_j \left[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j \right]
\end{aligned} \quad (26)$$

This hamiltonian is quadratic and diagonalized. Note that there are no other terms in the Hamiltonian. In general by reducing an interacting term in the Hamiltonian to a quadratic one the action of the unitary operator on other terms existing in the Hamiltonian is in general not trivial. Often, quadratic terms become now complex and in general interacting terms.

Similarly

$$\begin{aligned}
U_z H_X U_z^\dagger &= \frac{1}{4} \sum_j \left[U_z g_j^\dagger \gamma_{3,j} U_z^\dagger U_z \gamma_{3,j+1} g_{j+1} U_z^\dagger \right. \\
&\quad + U_z \gamma_{3,j} g_j U_z^\dagger U_z g_{j+1}^\dagger \gamma_{3,j+1} U_z^\dagger \\
&\quad + U_z g_j^\dagger \gamma_{3,j} U_z^\dagger U_z g_{j+1}^\dagger \gamma_{3,j+1} U_z^\dagger \\
&\quad \left. + U_z g_j \gamma_{3,j} U_z^\dagger U_z g_{j+1} \gamma_{3,j+1} U_z^\dagger \right]
\end{aligned} \quad (27)$$

Let us look at the last new two terms. These can be written as

$$-\frac{1}{4} \sum_j \left[g_j^\dagger g_{j+1}^\dagger + g_{j+1} g_j \right]$$

So,

$$H_X = - \sum_j \left[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j + g_j^\dagger g_{j+1}^\dagger + g_{j+1} g_j \right] \quad (28)$$

which is Kitaev's model for $t = 1, \Delta = 1$. Note that

$$U_{z,j} \left(1 - 2g_j^\dagger g_j \right) U_{z,j}^\dagger = 1 - 2g_j^\dagger g_j \quad (29)$$

The magnetic field term is invariant.

Let us now repeat the same steps but using a representation of the unitary operator fully in terms of Majoranas. We can see that

$$\begin{aligned}
U_{z,j} &= 1 - g_j^\dagger g_j + \gamma_{3,j} g_j^\dagger g_j \\
&= \frac{1}{2} [1 + i\gamma_{1,j} \gamma_{2,j} + \gamma_{3,j} - i\gamma_{1,j} \gamma_{2,j} \gamma_{3,j}]
\end{aligned} \quad (30)$$

Consider for instance

$$H_X = \frac{1}{4} \sum_j \gamma_{2,j} \gamma_{2,j+1} \gamma_{3,j} \gamma_{3,j+1} \quad (31)$$

This is a quartic Hamiltonian that we want to transform to a quadratic Hamiltonian once again by eliminating the

γ_3 terms. Note that the action of the unitary operator $U_{z,j}$ of the form

$$U_{z,j}\gamma_{2,j}\gamma_{3,j}U_{z,j}^\dagger = i\gamma_{1,j} \quad (32)$$

can be seen as transforming a spin-like operator (bilinear in the Majoranas) into a Majorana operator. We may also see that

$$U_{z,j}\gamma_{1,j}\gamma_{3,j}U_{z,j}^\dagger = -i\gamma_{2,j} \quad (33)$$

We can see that we obtain

$$\begin{aligned} U_z H_X U_z^\dagger &= -\frac{1}{4} \sum_j U_z \gamma_{2,j} \gamma_{3,j} U_z^\dagger U_z \gamma_{2,j+1} \gamma_{3,j+1} U_z^\dagger \\ &= -\frac{1}{4} \sum_j \left(\prod_{l=1}^{j-1} U_{z,l} \right) i\gamma_{1,j} \left(\prod_{l=j-1}^1 U_{z,l}^\dagger \right) \\ &\quad \left(\prod_{l=1}^j U_{z,l} \right) i\gamma_{1,j+1} \left(\prod_{l=j}^1 U_{z,l}^\dagger \right) \end{aligned} \quad (34)$$

Note now that, similarly to before,

$$\begin{aligned} \left(\prod_{l=1}^{j-1} U_{z,l} \right) \gamma_{1,j} \left(\prod_{l=j-1}^1 U_{z,l}^\dagger \right) &= \gamma_{1,j} \prod_{l=1}^{j-1} [i\gamma_{1,l}\gamma_{2,l}] \\ &= \gamma_{1,j} \prod_{l=1}^{j-1} (-2S_l^z) \end{aligned} \quad (35)$$

Once again the string operator appears. Finally we get that

$$U_z H_X U_z^\dagger = \frac{i}{4} \sum_j \gamma_{2,j} \gamma_{1,j+1} \quad (36)$$

V. UNITARY TRANSFORMATIONS

A. Majorana and spin operators

Being given the anticommutation relations $\{\gamma_i, \gamma_j\} = 2\delta_{ij}I$, where I is the identity, satisfied by the three Majorana operators γ_i , $i = 1, 2, 3$, one is naturally lead to define the spin operators $\mathcal{S}_i = \frac{-i}{2}\epsilon_{ijk}\gamma_j\gamma_k$, differing in normalization from the standard spin operators $\vec{S} = \frac{1}{2}\vec{\mathcal{S}}$, and the operator $I_3 = -i\gamma_1\gamma_2\gamma_3 = \frac{-i}{6}\epsilon_{ijk}\gamma_i\gamma_j\gamma_k$.

With the definition of these operators, the multiplication rule of the Majorana operators becomes

$$\gamma_i\gamma_j = \delta_{ij}I + i\epsilon_{ijk}\mathcal{S}_k \quad (37)$$

from which it follows that $\vec{\gamma}^2 = 3I$. The multiplication rules of these operators are given by

$$\gamma_i\mathcal{S}_j = \delta_{ij}I_3 + i\epsilon_{ijk}\gamma_k \quad (38)$$

$$\mathcal{S}_i\gamma_j = \delta_{ij}I_3 + i\epsilon_{ijk}\gamma_k \quad (39)$$

and

$$\mathcal{S}_i\mathcal{S}_j = \delta_{ij}I + i\epsilon_{ijk}\mathcal{S}_k \quad (40)$$

$$\mathcal{S}_i\mathcal{S}_j = \delta_{ij}I + i\epsilon_{ijk}\mathcal{S}_k \quad (41)$$

together with $I_3^2 = I$, $I_3\gamma_i = \gamma_i I_3 = \mathcal{S}_i$, and $I_3\mathcal{S}_i = \mathcal{S}_i I_3 = \gamma_i$. One also has $\vec{\mathcal{S}}^2 = 3I$ and $\vec{\gamma} \cdot \vec{\mathcal{S}} = \vec{\mathcal{S}} \cdot \vec{\gamma} = 3I_3$.

These equations can be summarized as that the operators γ_i and \mathcal{S}_i follow multiplication rules similar to the Pauli matrices, but paying attention to their nature, i.e. being odd or even in the Majorana operators, and that the multiplication by I_3 transforms γ_i into \mathcal{S}_i and vice versa.

Defining now the combinations $I_\pm = \frac{1}{2}(I \pm I_3)$, and $\Sigma_i^\pm = \frac{1}{2}(\mathcal{S}_i \pm \gamma_i)$ one finds that the product of two operators of different signs vanish and that

$$I_\pm^2 = I_\pm \quad (42)$$

$$\Sigma_i^+ \Sigma_j^+ = \delta_{ij}I_+ + i\epsilon_{ijk}\Sigma_k^+ \quad (43)$$

$$\Sigma_i^- \Sigma_j^- = \delta_{ij}I_- + i\epsilon_{ijk}\Sigma_k^- \quad (44)$$

i.e. they commute and satisfy two separate $SU(2)$ algebras, with I_\pm as the identities. One also has $(\vec{\Sigma}^\pm)^2 = 3I$ and $\vec{\Sigma}^+ \cdot \vec{\Sigma}^- = 0$

Additionally, besides the projectors I_\pm , it is also sometimes useful to use the operators $P_{\vec{n}}^{+\pm} = \frac{1}{2}(I_+ \pm \vec{n} \cdot \vec{\Sigma}^+)$ and $P_{\vec{n}}^{-\pm} = \frac{1}{2}(I_- \pm \vec{n} \cdot \vec{\Sigma}^-)$, which, for a given \vec{n} , are also projectors, i.e. they satisfy $P_a P_b = \delta_{ab} P_a$, with $a, b = \pm\pm$.

Finite operators involving projectors are generally given by

$$e^{i\alpha P} = I + (e^{i\alpha} - 1)P = (I - P) + e^{i\alpha}P \quad (45)$$

for a single operator, and

$$e^{i \sum_i \alpha_i P_i} = I + \sum_i (e^{i\alpha_i} - 1)P_i = (I - \sum_i P_i) + \sum_i e^{i\alpha_i} P_i \quad (46)$$

for several operators. When $\sum_i P_i = I$ the first term vanishes and one finds the usual decomposition of an exponential operator in its own complete basis, namely for the Boltzmann factor.

In particular, one has

$$e^{i\alpha I_\pm} = I + (e^{i\alpha} - 1)I_\pm = I_\mp + e^{i\alpha}I_\pm \quad (47)$$

The finite operators of the $SU(2)$ generators are given by

$$e^{i\frac{\theta}{2}\vec{n}\cdot\vec{\gamma}} = \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\gamma} \quad (48)$$

$$e^{i\frac{\theta}{2}\vec{n}\cdot\vec{\mathcal{J}}} = \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\mathcal{J}} \quad (49)$$

$$e^{i\frac{\theta}{2}\vec{n}\cdot\vec{\Sigma}_{\pm}} = I_{\mp} + \cos\frac{\theta}{2}I_{\pm} + i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\Sigma}_{\pm} \quad (50)$$

with \vec{n} a unitary vector, $\vec{n}^2 = 1$. In the last equation, the first term reflects the existence of two separate $SU(2)$ algebras, each with its own identity.

Finally,

$$e^{i\frac{\theta}{2}I_3} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}I_3 \quad (51)$$

More general operators can be obtained from these results, in particular using the last two equations and eq. (47), using $\vec{n}\cdot\vec{\mathcal{J}} = \vec{n}\cdot\vec{\Sigma}_+ + \vec{n}\cdot\vec{\Sigma}_-$, $\vec{n}\cdot\vec{\gamma} = \vec{n}\cdot\vec{\Sigma}_+ - \vec{n}\cdot\vec{\Sigma}_-$ and $I_3 = I_+ - I_-$.

The action of finite operations, namely of unitary operations, on the generators, are similar to the usual expression for the rotations of vectors, but paying due attention to the existence of the two $SU(2)$ commuting (and annihilating) algebras and to the parity of the different operators in terms of the Majorana operators, as patent in the next subsection.

B. Unitary and Hermitian operators

Let us consider general local transformations of the type $U^2 = I$, therefore unitary transformations that are also hermitian operators. Let us look for solutions of the type

$$U = zI + \omega I_3 + \vec{x}\cdot\vec{\gamma} + \vec{y}\cdot\vec{\mathcal{J}} \quad (52)$$

Here $I_3 = i\gamma_1\gamma_2\gamma_3$, $\vec{\mathcal{J}} = 2\vec{S}$ and $z, \omega, \vec{x}, \vec{y}$ are real numbers. These imply that U is unitary. The solutions of $U^2 = I$ may be organized into the following cases:

- (i) $U = I$
- (ii) $U = I_3$
- (iii) $U = \frac{1}{2}\left(I + I_3 + \vec{n}_- \cdot \frac{1}{2}(\vec{\mathcal{J}} - \vec{\gamma})\right)$, with $|\vec{n}_-| = 1$
- (iv) $U = \pm\frac{1}{2}\left(I - I_3 + \vec{n}_+ \cdot \frac{1}{2}(\vec{\mathcal{J}} + \vec{\gamma})\right)$, with $|\vec{n}_+| = 1$
- (v) $U = \cos\frac{\theta}{2}\vec{n}_1\cdot\vec{\gamma} + \sin\frac{\theta}{2}\vec{n}_2\cdot\vec{\mathcal{J}}$, with $|\vec{n}_1| = |\vec{n}_2| = 1$ and $\vec{n}_1\cdot\vec{n}_2 = 0$.

Let us summarize some relations. Both in classes (i) and (ii) we find that acting on operators on a given site

$$U\vec{\gamma}U^\dagger = \vec{\gamma} \quad U\vec{\mathcal{J}}U^\dagger = \vec{\mathcal{J}} \quad (53)$$

In the case of class (iii) we get

$$\begin{aligned} U\frac{1}{2}(\vec{\mathcal{J}} - \vec{\gamma})U^\dagger &= 2\vec{n}\left(\vec{n}\cdot\frac{1}{2}(\vec{\mathcal{J}} - \vec{\gamma})\right) - \frac{1}{2}(\vec{\mathcal{J}} - \vec{\gamma}) \\ U\vec{\gamma}U^\dagger &= \vec{\mathcal{J}} - \vec{n}\left(\vec{n}\cdot(\vec{\mathcal{J}} - \vec{\gamma})\right) \\ U\vec{\mathcal{J}}U^\dagger &= \vec{\gamma} + \vec{n}\left(\vec{n}\cdot(\vec{\mathcal{J}} - \vec{\gamma})\right) \end{aligned} \quad (54)$$

In the case of class (iv) we get that

$$\begin{aligned} U\vec{\gamma}U^\dagger &= -\vec{\mathcal{J}} + \vec{n}\left(\vec{n}\cdot(\vec{\mathcal{J}} + \vec{\gamma})\right) \\ U\vec{\mathcal{J}}U^\dagger &= -\vec{\gamma} + \vec{n}\left(\vec{n}\cdot(\vec{\mathcal{J}} + \vec{\gamma})\right) \end{aligned} \quad (55)$$

Finally, in the case of class (v) we get that

$$\begin{aligned} U\vec{\gamma}U^\dagger &= -\vec{\gamma} + 2\left[\left(\cos\frac{\theta}{2}\right)^2\vec{n}_1(\vec{n}_1\cdot\vec{\gamma}) \right. \\ &\quad + \left(\sin\frac{\theta}{2}\right)^2\vec{n}_2(\vec{n}_2\cdot\vec{\gamma}) \\ &\quad + \left.\sin\frac{\theta}{2}\cos\frac{\theta}{2}(\vec{n}_1(\vec{n}_2\cdot\vec{\gamma}) + \vec{n}_2(\vec{n}_1\cdot\vec{\gamma}))\right] \\ U\vec{\mathcal{J}}U^\dagger &= -\vec{\mathcal{J}} + 2\left[\left(\cos\frac{\theta}{2}\right)^2\vec{n}_1(\vec{n}_1\cdot\vec{\mathcal{J}}) \right. \\ &\quad + \left(\sin\frac{\theta}{2}\right)^2\vec{n}_2(\vec{n}_2\cdot\vec{\mathcal{J}}) \\ &\quad + \left.\sin\frac{\theta}{2}\cos\frac{\theta}{2}(\vec{n}_1(\vec{n}_2\cdot\vec{\mathcal{J}}) + \vec{n}_2(\vec{n}_1\cdot\vec{\mathcal{J}}))\right] \end{aligned} \quad (56)$$

C. General unitary operations

Let us now consider local transformations that are unitary but not hermitian. Some possible examples are illustrated next.

(a) $U = \frac{1}{\sqrt{2}}\left(1 + i\vec{n}\cdot\vec{\mathcal{J}}\right)$, with $|\vec{n}| = 1$. The action of this operator leads to

$$\begin{aligned} U\vec{\mathcal{J}}U^\dagger &= \vec{n}\left(\vec{n}\cdot\vec{\mathcal{J}}\right) + \vec{n}\times\vec{\mathcal{J}} \\ U\vec{\gamma}U^\dagger &= \vec{n}\left(\vec{n}\cdot\vec{\gamma}\right) + \vec{n}\times\vec{\gamma} \end{aligned} \quad (57)$$

The action of this unitary operator does not change the nature of the operators and therefore is an example of a canonical transformation.

(b) $U = \frac{1}{\sqrt{2}}(1 + i\vec{n}\cdot\vec{\gamma})$, with $|\vec{n}| = 1$. The action of this operator leads to

$$\begin{aligned} U\vec{\gamma}U^\dagger &= \vec{n}\left(\vec{n}\cdot\vec{\gamma}\right) + \vec{n}\times\vec{\mathcal{J}} \\ U\vec{\mathcal{J}}U^\dagger &= \vec{n}\left(\vec{n}\cdot\vec{\mathcal{J}}\right) + \vec{n}\times\vec{\gamma} \end{aligned} \quad (58)$$

The action of this unitary operator mixes the nature of the operators and therefore is an example of a non-canonical transformation.

(c) $U = \frac{1}{\sqrt{2}}(I_3 + i\vec{n} \cdot \vec{\gamma})$, with $|\vec{n}| = 1$ and $I_3 = -i\gamma_1\gamma_2\gamma_3$. The action of this operator leads to a set of canonical transformations

$$\begin{aligned} U\vec{\gamma}U^\dagger &= \vec{n}(\vec{n} \cdot \vec{\gamma}) + \vec{n} \times \vec{\gamma} \\ U\vec{\mathcal{S}}U^\dagger &= \vec{n}(\vec{n} \cdot \vec{\mathcal{S}}) + \vec{n} \times \vec{\mathcal{S}} \end{aligned} \quad (59)$$

(d) $U = \cos\theta/2 + i\sin\theta/2\vec{n} \cdot \vec{\gamma}$ leads to

$$\begin{aligned} U\vec{\gamma}U^\dagger &= \vec{n}(\vec{n} \cdot \vec{\gamma}) + \cos\theta(\vec{\gamma} - \vec{n}(\vec{n} \cdot \vec{\gamma})) \\ &\quad + \sin\theta(\vec{n} \times \vec{\gamma}) \\ U\vec{\mathcal{S}}U^\dagger &= \vec{n}(\vec{n} \cdot \vec{\mathcal{S}}) + \cos\theta(\vec{\mathcal{S}} - \vec{n}(\vec{n} \cdot \vec{\mathcal{S}})) \\ &\quad + \sin\theta(\vec{n} \times \vec{\mathcal{S}}) \end{aligned} \quad (60)$$

which is also non-canonical since it mixes the two types of operators, Majoranas and spin operators.

Note that from

$$U = \frac{1}{2}[I + I_3 - \vec{n} \cdot (\vec{\mathcal{S}} - \vec{\gamma})] \quad (61)$$

choosing $\vec{n} = \vec{e}_z$ we get that

$$\begin{aligned} U &= \frac{1}{2}[I + I_3 + \gamma_3 - \mathcal{S}_3] \\ &= \frac{1}{2}[1 - i\gamma_1\gamma_2\gamma_3 + \gamma_3 + i\gamma_1\gamma_2] = U_z \end{aligned} \quad (62)$$

is the operator U_z previously used to transform the spin chains into spinless fermions and diagonalize the problem. Also, note that

$$U_z\mathcal{S}_zU_z = \mathcal{S}_z \quad U_z\mathcal{S}_xU_z = \gamma_1 \quad U_z\mathcal{S}_yU_z = \gamma_2 \quad (63)$$

This class of transformations given by $\vec{n} = \vec{e}_\alpha$ leaves one of the spin operator components invariant:

$$U_\alpha\mathcal{S}_\alpha U_\alpha = \mathcal{S}_\alpha \quad (64)$$

for $\alpha = x, y, z(1, 2, 3)$. The transformation acts in the perpendicular plane.

Consider now the following unitary operator of class (v)

$$U_v = \cos\frac{\theta}{2}\vec{n}_1 \cdot \vec{\gamma} + \sin\frac{\theta}{2}\vec{n}_2 \cdot \vec{\mathcal{S}} = U_v^\dagger \quad (65)$$

with $\vec{n}_1 \cdot \vec{n}_2 = 0$. Take for instance $\vec{n}_1 = \vec{e}_x, \vec{n}_2 = \vec{e}_y$. Then

$$U_v = \cos\frac{\theta}{2}\gamma_1 + \sin\frac{\theta}{2}\mathcal{S}_2 = \cos\frac{\theta}{2}\gamma_1 - i\sin\frac{\theta}{2}\gamma_3\gamma_1 \quad (66)$$

The action of the operator on the spin components is

$$\begin{aligned} U_v\mathcal{S}_xU_v &= \cos\theta\mathcal{S}_1 + \sin\theta\gamma_2 \\ U_v\mathcal{S}_yU_v &= -\cos\theta\mathcal{S}_2 + \sin\theta\gamma_1 \end{aligned} \quad (67)$$

Also, taking two sites l and j

$$U_{v,l}\gamma_{\alpha,j}U_{v,l} = [-\cos\theta + i\sin\theta\gamma_{3,l}]\gamma_{\alpha,j} \quad (68)$$

with $\alpha = 1, 2, 3$.

The action of this operator on the XY spin model

$$\begin{aligned} H_{XY} &= \frac{J_X}{4} \sum_j \gamma_{2,j}\gamma_{3,j}\gamma_{2,j+1}\gamma_{3,j+1} \\ &\quad + \frac{J_Y}{4} \sum_j \gamma_{3,j}\gamma_{1,j}\gamma_{3,j+1}\gamma_{1,j+1} \end{aligned} \quad (69)$$

taking its product over all sites $U_v = \prod_{j=1}^N U_{v,j}$ gives

$$\begin{aligned} U_v H_{XY} U_v^\dagger &= \frac{J_X}{4} \sum_j [\cos\theta\gamma_{2,j}\gamma_{3,j} \\ &\quad + i\sin\theta \prod_{l=1}^{j-1} (-\cos\theta + i\sin\theta\gamma_{3,l}) \gamma_{2,j}] \\ &\quad [\cos\theta\gamma_{2,j+1}\gamma_{3,j+1} \\ &\quad + i\sin\theta \prod_{l=1}^j (-\cos\theta + i\sin\theta\gamma_{3,l}) \gamma_{2,j+1}] \\ &\quad + \frac{J_Y}{4} \sum_j [\cos\theta\gamma_{1,j}\gamma_{3,j} \\ &\quad + i\sin\theta \prod_{l=1}^{j-1} (-\cos\theta + i\sin\theta\gamma_{3,l}) \gamma_{1,j}] \\ &\quad [\cos\theta\gamma_{1,j+1}\gamma_{3,j+1} \\ &\quad + i\sin\theta \prod_{l=1}^j (-\cos\theta + i\sin\theta\gamma_{3,l}) \gamma_{1,j+1}] \end{aligned} \quad (70)$$

Choose $\theta = \pi/4$. In this case we get that

$$\begin{aligned} U H_{XY} U^\dagger &= \frac{iJ_X}{2} \sum_j \left(\prod_{l=1}^{j-2} (-1)^l \right) S_{j,x}\gamma_{2,j+1} \\ &\quad - \frac{iJ_Y}{2} \sum_j \left(\prod_{l=1}^{j-2} (-1)^l \right) S_{j,y}\gamma_{1,j+1} \end{aligned} \quad (71)$$

Note that now there are cubic terms in Majorana operators and so the quartic problem is not reduced to a quadratic problem. In other words this transformation converts the Hamiltonian into a product of Majorana and spin operators, which, however, in leading order does not conserve the fermionic number.

VI. HUBBARD MODEL

The Hubbard model is a fermionic interacting problem with an interaction of the type $Un_\uparrow n_\downarrow$. Let us introduce

Majorana representations for the fermionic operators

$$\begin{aligned} c_\uparrow &= \frac{1}{2} (\gamma_1^\uparrow - i\gamma_2^\uparrow) \\ c_\downarrow &= \frac{1}{2} (\gamma_1^\downarrow - i\gamma_2^\downarrow) \\ c_\uparrow^\dagger &= \frac{1}{2} (\gamma_1^\uparrow + i\gamma_2^\uparrow) \\ c_\downarrow^\dagger &= \frac{1}{2} (\gamma_1^\downarrow + i\gamma_2^\downarrow) \end{aligned} \quad (72)$$

The density operators can be written as

$$\begin{aligned} n_\uparrow &= \frac{1}{2} (1 - i\gamma_1^\uparrow \gamma_2^\uparrow) \\ n_\downarrow &= \frac{1}{2} (1 - i\gamma_1^\downarrow \gamma_2^\downarrow) \end{aligned} \quad (73)$$

Defining as usual $n = n_\uparrow + n_\downarrow$, we can get that

$$(n-1)^2 = \frac{1}{2} (1 - \gamma_1^\uparrow \gamma_2^\uparrow \gamma_1^\downarrow \gamma_2^\downarrow) \quad (74)$$

The interaction term can be written as

$$\frac{1}{2} U (n-1)^2 - \frac{U}{4} = U S_z^\uparrow S_z^\downarrow \quad (75)$$

Note that

$$\begin{aligned} n_\uparrow - \frac{1}{2} &= S_z^\uparrow = \left(\frac{-i}{2} \right) \gamma_1^\uparrow \gamma_2^\uparrow \\ n_\downarrow - \frac{1}{2} &= S_z^\downarrow = \left(\frac{-i}{2} \right) \gamma_1^\downarrow \gamma_2^\downarrow \end{aligned} \quad (76)$$

Defining the unitary operator

$$U_x^\alpha = \frac{1}{2} [1 - i\gamma_1^\alpha \gamma_2^\alpha \gamma_3^\alpha + \gamma_1^\alpha + i\gamma_2^\alpha \gamma_3^\alpha] \quad (77)$$

with $\alpha = \uparrow, \downarrow$. Recall that

$$U_x^\alpha S_z^\alpha (U_x^\alpha)^\dagger = \frac{1}{2} \gamma_3^\alpha \quad (78)$$

Define now

$$U_x = U_x^\uparrow U_x^\downarrow \quad (79)$$

Note that $U_x = U_x^\dagger$ and $U_x^2 = I$. The action of the operator on the interacting term can be understood from

$$\begin{aligned} U_x S_z^\uparrow S_z^\downarrow U_x^\dagger &= U_x^\uparrow U_x^\downarrow S_z^\uparrow S_z^\downarrow U_x^\downarrow U_x^\uparrow \\ &= \frac{\gamma_3^\uparrow}{2} U_x^\uparrow \frac{\gamma_3^\downarrow}{2} U_x^\uparrow \\ &= \frac{i}{4} \gamma_3^\uparrow \gamma_2^\uparrow \gamma_3^\uparrow \gamma_3^\downarrow \\ &= -\frac{i}{4} \gamma_2^\uparrow \gamma_3^\downarrow \end{aligned} \quad (80)$$

Defining

$$H_H^j = (S_z^\uparrow S_z^\downarrow)_j \quad (81)$$

the Hubbard interacting term can be written as

$$H_h = \sum_j H_H^j \quad (82)$$

To simplify notation let us just call $U = U_x$ and take the product over all lattice sites. We get then

$$\begin{aligned} U H_H U^\dagger &= \sum_j \left(\prod_{l=1}^N U_l^\dagger U_l^\downarrow \right) S_{z,j}^\uparrow S_{z,j}^\downarrow \left(\prod_{l=N}^1 U_l^\downarrow U_l^\uparrow \right) \\ &= -\frac{i}{4} \sum_j \left(\prod_{l=1}^{j-1} U_l^\dagger U_l^\downarrow \right) \gamma_{2,j}^\uparrow \gamma_{3,j}^\downarrow \left(\prod_{l=j-1}^1 U_l^\downarrow U_l^\uparrow \right) \\ &= -\frac{i}{4} \sum_j \gamma_{2,j}^\uparrow \gamma_{3,j}^\downarrow \end{aligned} \quad (83)$$

The transformation renders the interacting term into a quadratic term. However, its action on the kinetic term leads to a complicated form as we will see next. The kinetic term is given by

$$H_K = -t \sum_{j,\sigma} \left(c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma} \right) \quad (84)$$

with $\sigma = \uparrow, \downarrow$. In addition to the previous unitary operator $U = U_x$ we define an operator

$$\tilde{U} = \tilde{U}_x = \frac{1}{2} [1 + i\gamma_1 \gamma_2 \gamma_3 - \gamma_1 + i\gamma_2 \gamma_3] \quad (85)$$

Using that

$$\begin{aligned} U \gamma_1 \tilde{U} &= i\gamma_1 \gamma_2 \gamma_3 \\ U \gamma_2 \tilde{U} &= \gamma_1 \gamma_2 \\ U \tilde{U} &= i\gamma_2 \gamma_3 \end{aligned} \quad (86)$$

we get that

$$\begin{aligned} U H_K U^\dagger &= \sum_j \left(I_{3,j+1}^\uparrow I_{3,j}^\uparrow + I_{3,j}^\uparrow I_{3,j+1}^\uparrow + I_{3,j+1}^\downarrow I_{3,j}^\downarrow + I_{3,j}^\downarrow I_{3,j+1}^\downarrow \right) \\ &+ \sum_j \left(\mathcal{S}_{3,j+1}^\uparrow \mathcal{S}_{1,j}^\uparrow \mathcal{S}_{1,j}^\downarrow + \mathcal{S}_{1,j}^\downarrow \mathcal{S}_{1,j}^\uparrow \mathcal{S}_{3,j+1}^\uparrow \right. \\ &+ \left. \mathcal{S}_{1,j+1}^\uparrow \mathcal{S}_{3,j+1}^\downarrow \mathcal{S}_{1,j}^\downarrow + \mathcal{S}_{1,j}^\downarrow \mathcal{S}_{3,j+1}^\downarrow \mathcal{S}_{1,j+1}^\uparrow \right) \\ &+ \sum_j \left(\prod_{l=1}^{j-1} \mathcal{S}_{1,l}^\uparrow \mathcal{S}_{1,l}^\downarrow \right) \left(I_{3,j+1}^\uparrow \mathcal{S}_{3,j}^\uparrow + \mathcal{S}_{3,j}^\uparrow I_{3,j+1}^\uparrow \right. \\ &- \left. I_{3,j+1}^\downarrow \mathcal{S}_{3,j}^\downarrow \mathcal{S}_{1,j}^\uparrow + \mathcal{S}_{1,j}^\uparrow \mathcal{S}_{3,j}^\downarrow I_{3,j+1}^\downarrow \right) \end{aligned} \quad (87)$$

The first two terms have complex expressions involving six Majorana operators. The last one has a higher number of operators and in odd numbers. Assuming that any averages of these terms will vanish (to leading order)

we may consider in a first approximation only the even terms. Then we can write that

$$\begin{aligned}
UHU^\dagger = & -\frac{iU}{4} \sum_{j=1}^N \gamma_{2,j}^\uparrow \gamma_{3,j}^\downarrow \\
& - t \sum_{j,\sigma} (I_{3,j+1}^\sigma I_{3,j}^\sigma + I_{3,j}^\sigma I_{3,j+1}^\sigma) \\
& - t \sum_j \left(\mathcal{S}_{3,j+1}^\uparrow \mathcal{S}_{1,j}^\uparrow \mathcal{S}_{1,j}^\downarrow + \mathcal{S}_{1,j}^\downarrow \mathcal{S}_{1,j}^\uparrow \mathcal{S}_{3,j+1}^\uparrow \right. \\
& \left. + \mathcal{S}_{1,j+1}^\uparrow \mathcal{S}_{3,j+1}^\downarrow \mathcal{S}_{1,j}^\downarrow + \mathcal{S}_{1,j}^\downarrow \mathcal{S}_{3,j+1}^\downarrow \mathcal{S}_{1,j+1}^\uparrow \right)
\end{aligned} \tag{88}$$

One usual approach to simplify this term is to use a mean-field decoupling. This representation of the Hubbard model will be considered elsewhere.

The Hamiltonian obtained has a structure that is in some sense similar to the one obtained for instance in the exact transformation of Ref.². The original fermionic operators are transformed in that reference into new fermions called quasicharge and in new operators called quasispins (later decomposed into spins and pseudospins). The Hubbard- U term is reduced to a free term and the original hopping term becomes a term which quadratic in the quasicharges and a complicated term that couples the hoppings of the quasicharges with a quasispin-quasispin interaction term. Assuming a usual bilinear decomposition of the quasispin operators into bosons or fermions these complicated terms involve a power six of boson and fermion operators. The Hamiltonian obtained here is similar: the Hubbard- U term is quadratic but in terms of Majorana fermions of both species (\uparrow, \downarrow). The hopping involves a term that is bilinear in the I_3 operator. This is a Majorana-like operator and therefore we may understand this term as also a quadratic term in Majorana fermions. However, this operator is also given by a product of three Majoranas and therefore this hopping term may also be seen as a term with six operators. The remaining term involves three spin operators and therefore also has 6 operators in terms of the original Majoranas. Depending on the choice of independent variables we may need the six Majorana operators (three Majoranas for each spin component) or if we take the I_3 operator as an independent Majorana operator we have a further enlarged space of eight operators. Note that for instance $I_3 = \gamma_3 \mathcal{S}_z$.

VII. RELATION BETWEEN SPIN STATES OF XY MODEL AND FERMIONIC STATES

A. Kitaev model

Recall once again that the Kitaev model at the special point $t = \Delta$ reduces to eq. (13). If we define two non-

local operators²⁹

$$\begin{aligned}
d_j &= \frac{1}{2} (\gamma_{2,j} + i\gamma_{1,j+1}) \\
d_j^\dagger &= \frac{1}{2} (\gamma_{2,j} - i\gamma_{1,j+1})
\end{aligned} \tag{89}$$

these new operators are related to the other fermionic operators by

$$d_j^\dagger = \frac{1}{2} \left[-i(c_j - c_j^\dagger) - i(c_{j+1} + c_{j+1}^\dagger) \right] \tag{90}$$

We may note that

$$2d_j^\dagger d_j - 1 = i\gamma_{2,j} \gamma_{1,j+1} \tag{91}$$

Also

$$2d_j d_j^\dagger = 1 + (c_j^\dagger - c_j) (c_{j+1}^\dagger + c_{j+1}) \tag{92}$$

Define now the vacuum states of c operators in the usual way

$$c_j |0\rangle_j = 0 \tag{93}$$

for any site j . We may now define two states with different fermionic parities⁵⁵

$$\begin{aligned}
|I\rangle &= \left(\prod_{j=1}^{N-1} d_j d_j^\dagger \right) |0\rangle \\
|II\rangle &= \left(\prod_{j=1}^{N-1} d_j d_j^\dagger \right) c_N^\dagger |0\rangle
\end{aligned} \tag{94}$$

where

$$|0\rangle = |0\rangle_N \cdots |0\rangle_1 \tag{95}$$

States $|I\rangle$ and $|II\rangle$ have no excitations of the d operators. Let us analyse the state $|I\rangle$. The action on the vacuum is given by⁵⁵

$$\prod_{j=1}^{N-1} \left[1 + (c_j^\dagger - c_j) (c_{j+1}^\dagger + c_{j+1}) \right] |0\rangle = \prod_{j=1}^N (1 + c_j^\dagger) |0\rangle_{\text{even}} \tag{96}$$

where only even powers of c operators contribute. Similarly, considering the state $|II\rangle$ we find that⁵⁵

$$\prod_{j=1}^{N-1} \left[1 + (c_j^\dagger - c_j) (c_{j+1}^\dagger + c_{j+1}) \right] c_N^\dagger |0\rangle = \prod_{j=1}^N (1 + c_j^\dagger) |0\rangle_{\text{odd}} \tag{97}$$

These states are groundstates of the d operators and can be characterized by their fermionic parity $|I\rangle = |\psi_{\text{even}}^0\rangle$ and $|II\rangle = |\psi_{\text{odd}}^0\rangle$. We may now define two states that are linear combinations of these states as

$$\begin{aligned}
|\psi_0^\pm\rangle &= |\psi_{\text{even}}^0\rangle \pm |\psi_{\text{odd}}^0\rangle \\
|\psi_0^\pm\rangle &= \prod_{j=1}^N (1 \pm c_j^\dagger) |0\rangle
\end{aligned} \tag{98}$$

The groundstate of the d operators is the groundstate of the system, at this special point

$$|GS\rangle = |n_d = 0; n_d = 0; \dots; n_d = 0; \dots\rangle \quad (99)$$

written in terms of the d operators. The states $|\psi_0^\pm\rangle$ are the groundstates written in terms of the c operators.

B. Spin problem

Consider as an example the ferromagnetic X chain

$$H_X = - \sum_j S_{x,j} S_{x,j+1} \quad (100)$$

The groundstate is

$$|GS\rangle = |\sigma_x = 1; \sigma_x = 1; \dots; \sigma_x = 1; \dots\rangle \quad (101)$$

or $\sigma_x = -1$ at every site (Z_2 degeneracy). Recall that $\sigma_x = \mathcal{S}_x = -i\gamma_2\gamma_3$. Therefore the action of an operator at a given lattice site on the groundstate is

$$\begin{aligned} \mathcal{S}_x |\mathcal{S}_x = 1\rangle &= |\mathcal{S}_x = 1\rangle \\ (-i)\gamma_2\gamma_3 |GS\rangle &= |GS\rangle \end{aligned} \quad (102)$$

Define now new local fermionic operators in terms of the Majorana operators used to represent the spin operators as

$$\begin{aligned} f_j &= \frac{1}{2} (\gamma_{2,j} + i\gamma_{3,j}) \\ f_j^\dagger &= \frac{1}{2} (\gamma_{2,j} - i\gamma_{3,j}) \end{aligned} \quad (103)$$

Similarly to previous results it is easy to see that

$$2f_j^\dagger f_j - 1 = i\gamma_{2,j}\gamma_{3,j} \quad (104)$$

Therefore

$$\mathcal{S}_{1,j} = -i\gamma_{2,j}\gamma_{3,j} = -(2f_j^\dagger f_j - 1) \quad (105)$$

We may then establish an equivalence between the eigenstates of the spin operator and the eigenstates of the f operator as

$$\begin{aligned} \mathcal{S}_x |\mathcal{S}_x = 1\rangle &= |\mathcal{S}_x = 1\rangle \longleftrightarrow \\ &- (2f^\dagger f - 1) |0\rangle_f = |0\rangle_f \\ \mathcal{S}_x |\mathcal{S}_x = -1\rangle &= -|\mathcal{S}_x = -1\rangle \longleftrightarrow \\ &- (2f^\dagger f - 1) |1\rangle_f = -|1\rangle_f \end{aligned} \quad (106)$$

Therefore we can identify

$$\begin{aligned} |\mathcal{S}_x = 1\rangle &= |0\rangle_f \\ |\mathcal{S}_x = -1\rangle &= |1\rangle_f \end{aligned} \quad (107)$$

In other words the groundstate can be chosen as

$$|GS\rangle = |0; 0; \dots; 0; \dots\rangle_f \quad (108)$$

We can now see the influence of the U_z operator previously defined on the groundstate. One possible way to see this is to consider that when acting on a state with $n_f = 0$ the operator $-i\gamma_2\gamma_3 = I$, acts as the identity, I_j . Therefore in a trivial way we see that

$$(-i\gamma_2\gamma_3)_1 (-i\gamma_2\gamma_3)_2 \dots |0\rangle_f = |0\rangle_f \quad (109)$$

Acting with the unitary operator, U_z , we get

$$\begin{aligned} U_z |0\rangle_f &= \left(\prod_j U_{z,j} \right) |0\rangle_f \\ &= \prod_{j=1}^N \frac{1}{2} [\gamma_1 + i\gamma_2 + \gamma_1\gamma_3 - i\gamma_2\gamma_3]_j |0\rangle_f \end{aligned} \quad (110)$$

Using that $g^\dagger = (\gamma_1 + i\gamma_2)/2$ and $i\gamma_1\gamma_2 = 1 - 2g^\dagger g$, we find that

$$U_z |0\rangle_f = \prod_{j=1}^N [1 + g^\dagger - g^\dagger g]_j |0\rangle_f \quad (111)$$

Using that

$$|\mathcal{S}_x = 1\rangle \sim |\mathcal{S}_z = 1\rangle + |\mathcal{S}_z = -1\rangle \quad (112)$$

this implies that

$$|0\rangle_f \sim |0\rangle_g + |1\rangle_g \quad (113)$$

So,

$$U_z |GS\rangle \sim \prod_{j=1}^N [1 + g^\dagger]_j |0\rangle_g \sim |\psi_0^+\rangle \quad (114)$$

This last state was introduced in the context of the Kitaev model (like eq. (98)). So, the states are proportional, as expected. Using the unitary operator that transforms the spin Hamiltonian to the Kitaev spinless fermion model we also transform between the groundstates of the two models:

$$U_z |GS\rangle^{spins} \sim |GS\rangle_g^{Kitaev} \sim |GS\rangle_d \quad (115)$$

The operators g and f may be related by a unitary transformation that transforms \mathcal{S}_x into \mathcal{S}_z . Choosing a local operator as

$$\bar{U} = \frac{1}{\sqrt{2}} (1 - \gamma_1\gamma_3) \quad (116)$$

we can show that

$$\bar{U} f \bar{U}^\dagger = -ig^\dagger \quad \bar{U} f^\dagger \bar{U}^\dagger = ig \quad (117)$$

Its action on the spin operators may be determined and yields

$$\bar{U} \mathcal{S}_x \bar{U}^\dagger = \mathcal{S}_z \quad \bar{U} \mathcal{S}_y \bar{U}^\dagger = \mathcal{S}_y \quad \bar{U} \mathcal{S}_z \bar{U}^\dagger = -\mathcal{S}_x \quad (118)$$

Also,

$$\bar{U}\gamma_1\bar{U}^\dagger = \gamma_3 \quad \bar{U}\gamma_2\bar{U}^\dagger = \gamma_2 \quad \bar{U}\gamma_3\bar{U}^\dagger = -\gamma_1 \quad (119)$$

The state $|0\rangle_f$ may be expanded as

$$|0\rangle_f = \alpha|0\rangle_g + \beta|1\rangle_g \quad (120)$$

Acting with the creation operator f^\dagger we may obtain that

$$|1\rangle_f = -i(\alpha|1\rangle_g - \beta|0\rangle_g) \quad (121)$$

Imposing that $\langle 1|0\rangle_f = 0$ we get $\beta^*\alpha = \alpha^*\beta$, and using that $\langle \sigma_x = 1|\sigma_z|\sigma_x = 1\rangle = 0$ we obtain that $|\beta|^2 = |\alpha|^2$. Using that $|0\rangle_f$ is normalized $|\alpha|^2 + |\beta|^2 = 1$ we get that $\alpha = \beta = 1/\sqrt{2}$. In the Appendix we summarize the relation between the various states and the action of the Majoranas on them.

C. Order parameters

At zero temperature one may consider the operator $\sigma_x = -i\gamma_2\gamma_3$ as the order parameter of the ferromagnetic XX spin chain. The average value of this operator in a state where all the spins are oriented along the x direction is finite. As we have seen the operator may also be defined as $\sigma_x = -(2f^\dagger f - 1)$ and the groundstate is defined by selecting the occupation numbers of the f fermions as $n_f = f^\dagger f = 0$ at every lattice site. There is some liberty on calculating the average value of the order parameter within the framework of the XX chain. In addition to using the representation in terms of the f fermions we may use that

$$\sigma_x = -i\gamma_2\gamma_3 = -(g^\dagger - g)\gamma_3 \quad (122)$$

Then the matrix element of the order parameter in the groundstate simplifies to

$$\langle f|0|\sigma_x|0\rangle_f = -\langle f|0|(g^\dagger - g)\frac{f - f^\dagger}{i}|0\rangle_f \quad (123)$$

where we used that $\gamma_3 = -i(f - f^\dagger)$. Expanding the state $|0\rangle_f$ in terms of the states $|0\rangle_g$ and $|1\rangle_g$ it is easy to show that $\langle f|0|\sigma_x|0\rangle_f = 1$.

This order parameter of the spin chain, for which there is Landau type order and no topology, has a dual in the topological Kitaev model. This can be obtained performing the same unitary transformation, U_z , that is used to perform the duality transformation of the Hamiltonians. The dual operator is therefore defined as

$$U_z\sigma_{x,j}U_z^\dagger = U_z(-i\gamma_{2,j}\gamma_{3,j})U_z^\dagger \quad (124)$$

Using the results of eq. (25) we obtain that

$$U_z\sigma_{x,j}U_z^\dagger = (-1)^{\sum_{l=1}^{j-1} g_l^\dagger g_l} (g_j^\dagger + g_j) \quad (125)$$

The eigenstates of the Kitaev model at the point that corresponds to the XX chain ($\mu = 0, t = \Delta$), are better expressed in terms of the operators d_j, d_j^\dagger . We get therefore that

$$U_z\sigma_{x,j}U_z^\dagger = (-1)^{\sum_{l=1}^{j-1} g_l^\dagger g_l} i (d_{j-1}^\dagger - d_{j-1}) \quad (126)$$

Using that

$$(1 - 2g_{j-1}^\dagger g_{j-1}) (d_{j-1}^\dagger - d_{j-1}) = -(d_j + d_j^\dagger) \quad (127)$$

we get that

$$U_z\sigma_{x,j}U_z^\dagger = -i(-1)^{\sum_{l=1}^{j-2} g_l^\dagger g_l} (d_j^\dagger + d_j) \quad (128)$$

and therefore its average value vanishes. As discussed before⁵⁵ the dual operators do not follow, in general, directly from standard order parameters even though the procedure may provide interesting information on order parameters in topological phases using as starting points order parameters in Landau like systems³⁹. Recent work on possible order parameters in topological phases using the reduced density matrix has been proposed⁵⁶.

VIII. TOPOLOGICAL INVARIANTS

Let us focus now on the topological properties of the fermionic problem. One way to determine the topological properties of Kitaev's model is to consider the winding number.

A. Winding number: chiral symmetry

In momentum space we may write the Kitaev model as

$$\hat{H} = \frac{1}{2} \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \quad (129)$$

where

$$H_k^c = \begin{pmatrix} \epsilon_k - \mu & -2i\Delta \sin k \\ 2i\Delta \sin k & -\epsilon_k + \mu \end{pmatrix} \quad (130)$$

with $\epsilon_k = -2t \cos k$. Here c_k is the Fourier transform of c_j . A criterium for topology is the winding number. Consider a chiral symmetry operator C such that $CHC^\dagger = -H$. In our case $C = \tau_x$, where τ_x is a Pauli matrix. Defining a matrix T with columns the eigenvectors of $C = \tau_x$ one can show that

$$THT^\dagger = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix} \quad (131)$$

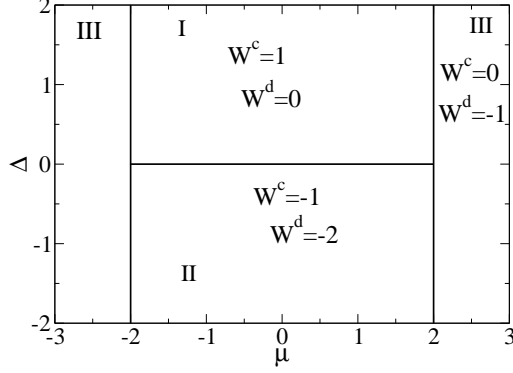


FIG. 1: Winding numbers W^c and W^d of the various phases. The winding number respecting to the Hamiltonian in the representation of the \tilde{d} operators takes the values $W^{\tilde{d}} = 0, -2, -1$ in phases *II*, *I*, *III*, respectively.

with $q(k) = \epsilon_k - \mu + 2i\Delta \sin k$. The winding number is defined as

$$W = \frac{1}{4\pi i} \int_{-\pi}^{\pi} dk \text{Tr} \left[q^{-1} \frac{dq(k)}{dk} - (q^\dagger)^{-1} \frac{dq^\dagger(k)}{dk} \right] \quad (132)$$

Calculating W we get that in region *I* of the phase diagram $W = 1$, in region *II* we get $W = -1$ and in the trivial phases we get $W = 0$. As is well known, W counts the number of protected edge modes on each edge of the chain. Also its sign depends on how one winds around the origin of the Brillouin zone.

Let us now consider the substitution eq. (90)

$$\begin{aligned} c_j &= \frac{i}{2} \left[d_{j-1}^\dagger - d_{j-1} + d_j + d_j^\dagger \right] \\ c_j^\dagger &= -\frac{i}{2} \left[d_{j-1} - d_{j-1}^\dagger + d_j + d_j^\dagger \right] \end{aligned} \quad (133)$$

These operators are specially useful at $\mu = 0, \Delta = t$, but now we want to use them everywhere in the phase diagram. The Hamiltonian becomes more complicated. Kitaev's model in terms of these d_j operators is

$$\begin{aligned} H &= \frac{1}{2} \sum_j (-t + \Delta) \left(-d_{j+1}^\dagger d_{j-1} - d_{j-1}^\dagger d_{j+1} \right. \\ &\quad \left. + d_{j-1} d_{j+1} + d_{j+1}^\dagger d_{j-1}^\dagger \right) \\ &\quad + \frac{1}{2} \sum_j (t + \Delta) \left(2d_j^\dagger d_j - 1 \right) \\ &\quad - \frac{\mu}{2} \sum_j \left[-d_{j-1}^\dagger d_j - d_j^\dagger d_{j-1} + d_{j-1} d_j + d_j^\dagger d_{j-1}^\dagger \right] \end{aligned} \quad (134)$$

When $\mu = 0, \Delta = t$ we get back the simple Hamiltonian. The chemical potential term now has nearest-neighbor terms both in the hopping and pairing. The simple Hamiltonian is like a chemical potential term (local in space). There is now a term with second-neighbors both in hopping and in pairing.

We get that the Hamiltonian in momentum space is given by

$$\hat{H} = \frac{1}{2} \sum_k \begin{pmatrix} d_k^\dagger & d_{-k} \end{pmatrix} H_k^d \begin{pmatrix} d_k \\ d_{-k}^\dagger \end{pmatrix} \quad (135)$$

where

$$H_k^d = \begin{pmatrix} \mu \cos k + (t + \Delta) & i\mu \sin k + i(t - \Delta) \sin 2k \\ -i\mu \sin k + i(-t + \Delta) \sin 2k & -\mu \cos k - (t + \Delta) - (t - \Delta) \cos 2k \end{pmatrix} \quad (136)$$

This is of the type of the Kitaev model with second-neighbors.

The matrix H_k^d allows us to perform a chiral transformation which leads to

$$\begin{aligned} q^d(k) &= -(-t + \Delta) \cos 2k + (t + \Delta) + \mu \cos k \\ &\quad + i(-t + \Delta) \sin 2k - i\mu \sin k \end{aligned} \quad (137)$$

Consider for instance the special point $\mu = 0, \Delta = t$. The

winding number

$$W^d = W^d(q^d(k)) = 0 \quad (138)$$

So in terms of the d operators the winding number in the special point indicates a trivial phase. For instance at $\Delta = t, \mu > 2t$ we get $W^d = 1$. So it appears that from the point of view of q^d versus q^c the model is “dual” and an apparent change of topology takes place.

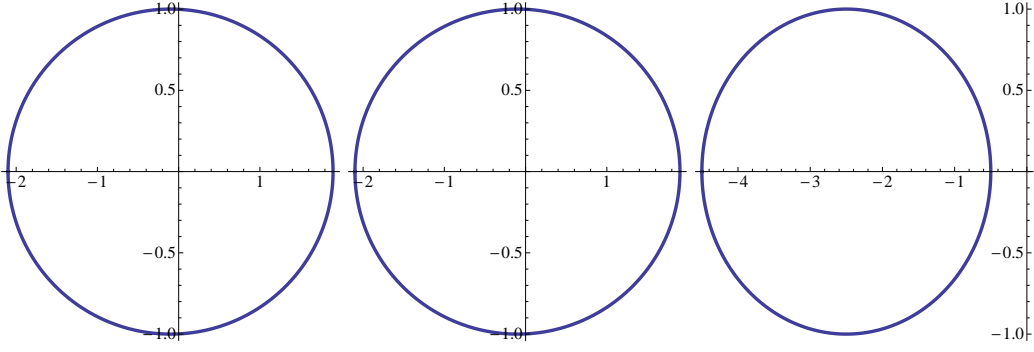


FIG. 2: Parametric plots of vector \vec{h}_c for i) phase I (left column), ii) phase II (center column) and iii) phase III (right column). The results are consistent with those for the winding number found in Fig. 1.

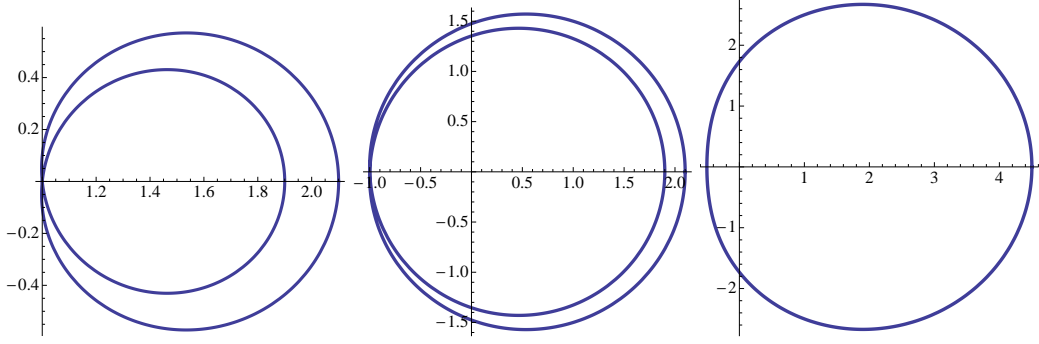


FIG. 3: Parametric plots of vector \vec{h}_d for i) phase I (left column), ii) phase II (center column) and iii) phase III (right column).

Since in terms of the d -operators the Hamiltonian has first and second neighbors, it is actually like an effective Kitaev model with first and second neighbors. But in this case the model has a phase diagram which is richer with winding numbers $W = 0, \pm 1, \pm 2$. The model is in the BDI class with a Z invariant. For instance, considering $\mu = 0$, we can show that it is equivalent to an effective Kitaev model with parameters $(\mu_2, t_2, t'_2, \Delta_2, \Delta'_e)$ where $\mu = 0$ implies that $t_2 = \Delta_2 = 0$ (no nearest-neighbor couplings, being hoppings or pairings). It is well-known that this model has two Majoranas at each edge. With no nearest neighbor couplings the model is like two decoupled chains and so the number of edge modes just doubles.

At point $\mu = 0, \Delta = t$ we diagonalized the Kitaev Hamiltonian in real space using Majorana operators and then introducing new fermionic operators. We may achieve something similar at the point $\mu = 0, \Delta = -t$. At this point the fermionic operator that diagonalizes the Hamiltonian may be defined as $\tilde{d}_j = \frac{-i}{2}(\gamma_{1,j} - i\gamma_{2,j+1})$. This new operator allows to rewrite the Hamiltonian as

$$H = t \sum_j \left(2\tilde{d}_j^\dagger \tilde{d}_j - 1 \right) \quad (139)$$

the groundstate is obtained taking at each site $\tilde{d}_j^\dagger \tilde{d}_j = 0$.

We get the relation

$$\tilde{d}_j = \frac{i}{2} \left[c_j + c_j^\dagger + c_{j+1}^\dagger - c_{j+1} \right] \quad (140)$$

Replacing the operators c_j, c_j^\dagger by the operators $\tilde{d}_j, \tilde{d}_j^\dagger$ at an arbitrary point in the phase diagram, we get a similar expression just replacing $\Delta \rightarrow -\Delta$. We get that in the original topological phase the winding number gives also zero. In Fig. 1 we compare the winding number using the various representations.

B. Winding of Hamiltonian

A related way that allows to determine the topological properties is obtained considering the Hamiltonian operator itself, instead of analysing the winding number. Note that we can write that $H_k^c = \vec{h}_c(k) \cdot \vec{\sigma}$, $H_k^d = \vec{h}_d(k) \cdot \vec{\sigma}$, $H_k^{\tilde{d}} = \vec{h}_{\tilde{d}}(k) \cdot \vec{\sigma}$. The vectors have no x -component and are given by

$$\begin{aligned} h_{c,z} &= \epsilon_k - \mu, h_{c,y} = 2\Delta \sin k \\ h_{d,z} &= \mu \cos k + (t + \Delta) + (t - \Delta) \cos 2k, h_{d,y} \\ &= \mu \sin k + (t - \Delta) \sin 2k \\ h_{\tilde{d},z} &= \mu \cos k + (t - \Delta) + (t + \Delta) \cos 2k, h_{\tilde{d},y} \\ &= \mu \sin k + (t + \Delta) \sin 2k \end{aligned} \quad (141)$$

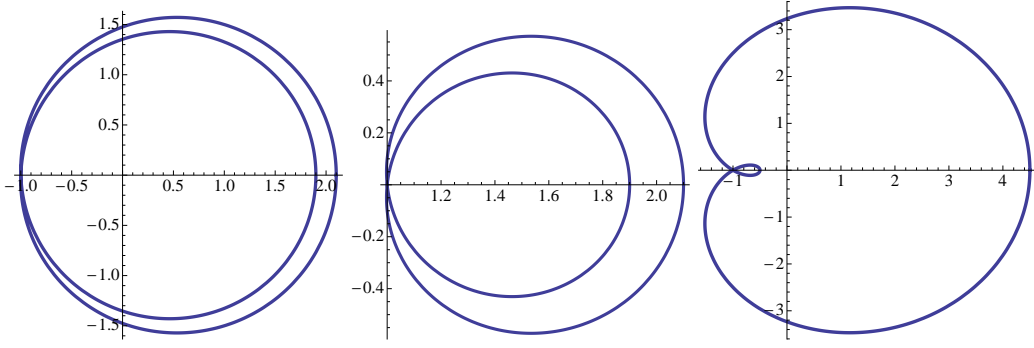


FIG. 4: Parametric plots of vector \vec{h}_d for i) phase I (left column), ii) phase II (center column) and iii) phase III (right column).

If we follow the two-dimensional parametric plot of the vectors \vec{h} on different parts of the phase diagram we get that distinct topologies are obtained if the origin ($\vec{h} = 0$) is included or not. The results are shown in Figs. (2,3,4). In phase I we get that \vec{h}_c and $\vec{h}_{\bar{d}}$ include the origin but \vec{h}_d does not. In phase II the roles of \vec{h}_d and $\vec{h}_{\bar{d}}$ are reversed. In the trivial regions III only the vector \vec{h}_c does not include the origin. These results lead to the same type of conclusion as that obtained from the comparison of the winding numbers.

C. Parity of groundstate

Another criterium that provides information about the topological properties of the system is given by the parity of the groundstate. Recall that the Kitaev model at the point $\mu = 0, \Delta = t$ can be written as

$$H = t \sum_{j=1}^{N-1} (2d_j^\dagger d_j - 1) \quad (142)$$

where the operator d_j is defined in eq. (89). If we use open boundary conditions (OBC) we see immediately that $d_N^\dagger d_N$ does not appear in the Hamiltonian, which leads to a degenerate groundstate ($d_N^\dagger d_N = 0, 1$) with two Majoranas.

If we use periodic boundary conditions (PBC) we just need to add this term ($d_N^\dagger d_N$):

$$H = t \sum_{j=1}^N (2d_j^\dagger d_j - 1) \quad (143)$$

Note that $d_N = \frac{i}{2}(c_N^\dagger - c_N + c_1 + c_1^\dagger)$. Therefore, $d_N^\dagger + d_N = i(c_N^\dagger - c_N)$ and $d_N - d_N^\dagger = i(c_1 + c_1^\dagger)$. Recall

$$\begin{aligned} |\psi_0^{even}\rangle &= \prod_{j=1}^N (1 + c_j^\dagger)_{even} |0\rangle_c \\ |\psi_0^{odd}\rangle &= \prod_{j=1}^N (1 + c_j^\dagger)_{odd} |0\rangle_c \end{aligned} \quad (144)$$

Then

$$\begin{aligned} (d_N + d_N^\dagger) |\psi_0^{even}\rangle &= i |\psi_0^{odd}\rangle \\ (d_N + d_N^\dagger) |\psi_0^{odd}\rangle &= -i |\psi_0^{even}\rangle \\ (d_N - d_N^\dagger) |\psi_0^{odd}\rangle &= i |\psi_0^{even}\rangle \\ (d_N - d_N^\dagger) |\psi_0^{even}\rangle &= i |\psi_0^{odd}\rangle \end{aligned} \quad (145)$$

This implies that⁵⁵

$$d_N |\psi_0^{odd}\rangle = 0 \quad (146)$$

Also

$$d_N^\dagger |\psi_0^{odd}\rangle = -i |\psi_0^{even}\rangle \quad (147)$$

We can conclude that the groundstate is non-degenerate and has odd parity. In the topological region parity is odd and in the trivial region parity is even. This is similar to the well-known result that the Majorana number being minus one or one signals the presence or absence of edge states²⁹. Note that the parity is preserved independently of the representation used.

D. Berry phase

Let us now calculate the Berry phase of the eigenstates of the Hamiltonian using different representations of the Hamiltonian of the system and therefore different states basis. Performing a Fourier transform of the relation between the operators c and d in real space we may show that

$$\begin{pmatrix} d_k \\ d_{-k}^\dagger \end{pmatrix} = U_k^\dagger \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \quad (148)$$

with

$$U_k = -e^{-\frac{ik}{2}} \begin{pmatrix} \sin \frac{k}{2} & i \cos \frac{k}{2} \\ -i \cos \frac{k}{2} & -\sin \frac{k}{2} \end{pmatrix} \quad (149)$$

which implies that $H_k^d = U_k^\dagger H_k^c U_k$. Performing the change from c_k to d_k corresponds to diagonalizing the problem at $\mu = 0, \Delta = t$ and performing the change from c_k to d_k corresponds to diagonalizing the problem at $\mu = 0, \Delta = -t$. At the points $\mu = 0, \Delta = \pm t$ the eigenvalues are ± 2 as we have seen before and the eigenvectors are

$$\psi_+ = \text{sgn} \left[\cos \frac{k}{2} \right] \begin{pmatrix} -i \frac{\Delta}{t} \sin \frac{k}{2} \\ \cos \frac{k}{2} \end{pmatrix} \quad (150)$$

and

$$\psi_- = \text{sgn} \left[\cos \frac{k}{2} \right] \begin{pmatrix} \cos \frac{k}{2} \\ -i \frac{\Delta}{t} \sin \frac{k}{2} \end{pmatrix} \quad (151)$$

Taking now $\mu = 0$ but any value of Δ , the eigenvalues are

$$\lambda_{\pm} = \pm 2 \sqrt{(t \cos k)^2 + (\Delta \sin k)^2} \quad (152)$$

The eigenvectors are

$$\psi_+^\Delta = \begin{pmatrix} \frac{-i2\Delta \sin k}{\sqrt{2\lambda_+ (\lambda_+ + 2t \cos k)}} \\ \sqrt{\frac{\lambda_+ + 2t \cos k}{2\lambda_+}} \end{pmatrix} \quad (153)$$

$$\psi_-^\Delta = \begin{pmatrix} \sqrt{\frac{\lambda_- - 2t \cos k}{2\lambda_-}} \\ \frac{-i2\Delta \sin k}{\sqrt{2\lambda_- (\lambda_- - 2t \cos k)}} \end{pmatrix} \quad (154)$$

Let us now consider the Berry phase in momentum space (Zak phase). For a given eigenstate n is given by

$$\gamma_n = \frac{i}{\pi} \int dk \langle \psi_n(k) | \frac{\partial}{\partial k} | \psi_n(k) \rangle \quad (155)$$

As we change variables (or operator descriptions) the states also change. Consider one abelian change:

$$|\psi_n(k)\rangle \rightarrow e^{i\xi(k)} |\psi_n(k)\rangle \quad (156)$$

Then we get that

$$\begin{aligned} \tilde{\gamma}_n &= \frac{i}{\pi} \int dk \langle \psi_n(k) | e^{-i\xi(k)} \frac{\partial}{\partial k} [e^{i\xi(k)} |\psi_n(k)\rangle] \\ &= \frac{i}{\pi} \int dk \langle \psi_n(k) | \frac{\partial}{\partial k} |\psi_n(k)\rangle - \frac{1}{\pi} \int dk \frac{d\xi(k)}{dk} \\ &= \gamma_n + \delta\gamma_n \end{aligned} \quad (157)$$

If the function $\xi(k)$ is periodic the Zak phase is invariant. This is similar to the well known case of adiabatic transport of some Hamiltonian that depends on some parameter and one considers a cyclic transport: in this case the Berry phase is invariant and observable and is related

to the polarization of a system of charges⁵⁷. In our case the state is a vector and in general we have a transformation from $|\psi_n\rangle$ to $|\tilde{\psi}_n\rangle$:

$$\begin{aligned} |\tilde{\psi}_n\rangle &= U^\dagger |\psi_n\rangle \\ \langle \tilde{\psi}_n| &= \langle \psi_n| U \end{aligned} \quad (158)$$

and

$$\tilde{H} = U^\dagger H U \quad (159)$$

Then we define

$$\begin{aligned} \gamma_n &= \frac{i}{\pi} \int dk \langle \psi_n | \partial_k | \psi_n \rangle \\ \tilde{\gamma}_n &= \frac{i}{\pi} \int dk \langle \tilde{\psi}_n | \partial_k | \tilde{\psi}_n \rangle \\ &= \gamma_n + \delta\gamma_n \end{aligned} \quad (160)$$

where

$$\delta\gamma_n = \frac{i}{\pi} \int dk \langle \psi_n | U (\partial_k U^\dagger) | \psi_n \rangle \quad (161)$$

In general $\delta\gamma_n \neq 0$. We can see that

$$\begin{aligned} \gamma_n &= \tilde{\gamma}_n + \frac{i}{\pi} \int dk \langle \tilde{\psi}_n | U^\dagger (\partial_k U) | \tilde{\psi}_n \rangle \\ \tilde{\gamma}_n &= \gamma_n + \frac{i}{\pi} \int dk \langle \psi_n | U (\partial_k U^\dagger) | \psi_n \rangle \end{aligned} \quad (162)$$

Defining a new phase as

$$\tilde{\Gamma}_n = \frac{i}{\pi} \int dk \langle \tilde{\psi}_n | D_k | \tilde{\psi}_n \rangle \quad (163)$$

with

$$D_k = \partial_k - (\partial_k U^\dagger) U \quad (164)$$

this new phase is invariant in the sense that $\Gamma_n = \tilde{\Gamma}_n$. The differential operator D_k is similar to a covariant derivative and similar to a non-abelian gauge transformation, that is needed if states are degenerate, which is not the case here.

Let us now calculate the Zak phase of the state $|\psi_-\rangle_c$. This is given by

$$\begin{aligned} \gamma_{-1} &= \frac{i}{\pi} \int_0^{2\pi} dk \langle \psi_- | \partial_k | \psi_- \rangle_c \\ &= \frac{i}{\pi} \int_0^{2\pi} dk \frac{d}{dk} \left(\ln \text{sgn} \left[\cos \frac{k}{2} \right] \right) \\ &= -1 = \gamma_c \end{aligned} \quad (165)$$

Only the singular part of the wave functions contributes.

Calculate now the Zak phase of the lowest eigenstate of $H_k^d (\mu = 0, \Delta = t)$. This state is simply

$$|\psi_-\rangle_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (166)$$

and

$$\gamma_{-1} = \gamma_d = 0 \quad (167)$$

Using that

$$\gamma_d = \gamma_c + \delta\gamma \quad (168)$$

and calculating $\delta\gamma$ we get $\delta\gamma = 1$, showing that the change of topology is hidden in the transformation, as expected. Similar results occur in other topological models such as the topologically non-trivial Shockley model.

1. Singular vs. non-singular transformations

Consider now a transformation between two points in parameter space. For instance, two points at $\mu = 0$ but with different values of Δ . Define diagonalized Hamiltonians

$$\begin{aligned} H_d^\Delta &= U_\Delta^\dagger H_\Delta U_\Delta \\ H_d^{\Delta'} &= U_{\Delta'}^\dagger H_{\Delta'} U_{\Delta'} \end{aligned} \quad (169)$$

The eigenvalues are of the form

$$\begin{aligned} \lambda_\Delta &= 2\sqrt{(t \cos k)^2 + (\Delta \sin k)^2} \\ \lambda_{\Delta'} &= 2\sqrt{(t \cos k)^2 + (\Delta' \sin k)^2} \end{aligned} \quad (170)$$

Then $H_d^{\Delta'} = (\lambda_{\Delta'}/\lambda_\Delta) H_d^\Delta$ which implies that

$$H_{\Delta'} = \frac{\lambda_{\Delta'}}{\lambda_\Delta} \mathcal{U}^\dagger H_\Delta \mathcal{U} \quad (171)$$

Here $\mathcal{U} = U_\Delta U_{\Delta'}^\dagger$. We can relate the eigenstates of the two Hamiltonians defined by

$$\begin{aligned} H_\Delta |\psi_\Delta\rangle &= \lambda_\Delta |\psi_\Delta\rangle \\ H_{\Delta'} |\psi_{\Delta'}\rangle &= \lambda_{\Delta'} |\psi_{\Delta'}\rangle \end{aligned} \quad (172)$$

as

$$|\psi_\Delta\rangle = \mathcal{U} |\psi_{\Delta'}\rangle \quad (173)$$

The Berry phases may be calculated as

$$\begin{aligned} \gamma_\Delta &= \frac{i}{\pi} \int_0^{2\pi} dk \langle \psi_\Delta | \partial_k | \psi_\Delta \rangle \\ \gamma_{\Delta'} &= \frac{i}{\pi} \int_0^{2\pi} dk \langle \psi_{\Delta'} | \partial_k | \psi_{\Delta'} \rangle \end{aligned} \quad (174)$$

As shown above they are related by

$$\gamma_{\Delta'} = \gamma_\Delta + \frac{i}{\pi} \int_0^{2\pi} dk \langle \psi_\Delta | \mathcal{U} (\partial_k \mathcal{U}^\dagger) | \psi_\Delta \rangle \quad (175)$$

Consider as an example, $\Delta = 1, \Delta' = -1$. In these simple cases $\lambda_\Delta = \lambda_{\Delta'} = 1$. The operator is simply given by

$$\mathcal{U} = \frac{1}{2} \begin{pmatrix} 2 \cos k & 2i \sin k \\ 2i \sin k & 2 \cos k \end{pmatrix} \quad (176)$$

Note that $1 + \cos k \geq 0$ and no singular part appears. This suggests that $\delta\gamma = 0$. Indeed using that

$$|\psi_-\rangle(\Delta = 1) = \text{sgn} \left[\cos \frac{k}{2} \right] \begin{pmatrix} \cos \frac{k}{2} \\ i \sin \frac{k}{2} \end{pmatrix} \quad (177)$$

and

$$|\psi_-\rangle(\Delta = -1) = \text{sgn} \left[\cos \frac{k}{2} \right] \begin{pmatrix} \cos \frac{k}{2} \\ -i \sin \frac{k}{2} \end{pmatrix} \quad (178)$$

we get that $\gamma(\Delta = -1) = -1$, which is the same as for $\Delta = 1$.

We may also change the parameters from a topological to a trivial phase. For instance we may consider the point, $A, \mu = 0, \Delta = t$ and the point, $B, \mu > 2t, \Delta = 0$. The Hamiltonian at this trivial point is diagonal. So the operator that diagonalizes it is the identity. Therefore $\mathcal{U} = U_{\Delta=t}$ and as we have seen before $U_{\Delta=1}$ is singular. Therefore $\gamma_B = \gamma_A + \delta\gamma = -1 + 1 = 0$. Note that the winding number distinguishes the phases *I* and *II* in the phase diagram. It seems that the Berry phase does not. However, $\gamma = -1 = -\pi/\pi$ is the same as $\gamma = 1$ since they differ by $2\pi/\pi$.

E. Interacting spin model

The main point is however the change of topology as one transforms from the spin problem to the fermionic dual problem. The spin-1/2 X model given by eq. (15) can be represented in terms of Majorana operators as in eq. (18). As established previously³⁸ the spin model is topologically trivial. A difficulty however arises since it is an interacting problem when the Majorana representation is used. Different conventional fermionic representations can be constructed from the Majorana fermions. One possibility is to define a local transformation as $f_j = (\gamma_{2,j} + i\gamma_{3,j})/2$ which implies that we can write $i\gamma_{2,j}\gamma_{3,j} = 2f_j^\dagger f_j - 1$. In terms of these operators we can write that

$$H_X = \sum_j \left(f_j^\dagger f_j - \frac{1}{2} \right) \left(f_{j+1}^\dagger f_{j+1} - \frac{1}{2} \right) \quad (179)$$

This is an interacting problem and its topological properties may be obtained imposing twisted boundary conditions and calculate the Berry phase averaging over the twist angle^{53,56}. However, in this representation, since all terms are of the type of density operators, there is no dependence on the twist angle and the Berry phase vanishes, as expected of a topologically trivial system. The same type of result was obtained before: the diagonalization of Kitaev's model at the point $\mu = 0, \Delta = t$ also allows to write the Hamiltonian in terms of the density operator and therefore the topological invariant vanishes in the same manner.

However, there is freedom to choose other fermionic representations in terms of the Majorana operators. For instance, the local representation used before in eq. (17),

leads to eq. (19) with terms that are non-local and therefore an explicit numerical calculation of the Berry phase is required. Eliminating the γ_3 operators leads to a topologically non-trivial problem, as elaborated on before.

Also, we may choose a non-local transformation of the type $h_j = (\gamma_{2,j} + i\gamma_{3,j+1})/2$. This leads to $i\gamma_{2,j}\gamma_{3,j+1} = 2h_j^\dagger h_j - 1$ and the Hamiltonian may be written as

$$H_X = \frac{1}{4} \sum_j \left(2h_j^\dagger h_j - 1 \right) \left(h_{j-1} h_{j+1} + h_{j-1} h_{j+1}^\dagger - h_{j-1}^\dagger h_{j+1} - h_{j-1}^\dagger h_{j+1}^\dagger \right) \quad (180)$$

which is of the type of correlated hoppings and pairings and that also requires an explicit calculation of the topological invariant. Note that the two sets of fermionic operators are related by a transformation that is non-trivial and may lead to a change of topology as above for the non-interacting (quadratic) problem. In this case, since the Hamiltonian is interacting the states are not straightforwardly obtained and the change of the Berry phase involves now a trace over a set of eigenstates that has to be obtained numerically. The relation between the operators h and f in momentum space is given by

$$\begin{pmatrix} h_k \\ h_{-k}^\dagger \end{pmatrix} = U_k^\dagger \begin{pmatrix} f_k \\ f_{-k}^\dagger \end{pmatrix} \quad (181)$$

with

$$U_k = e^{-\frac{ik}{2}} \begin{pmatrix} \cos \frac{k}{2} & i \sin \frac{k}{2} \\ i \sin \frac{k}{2} & \cos \frac{k}{2} \end{pmatrix} \quad (182)$$

This transformation is similar to the singular transformations considered in subsection 8.4. This raises the question of possible different topological numbers of the interacting problem, depending on the fermionic representation. This may occur prior to the unitary transformation that reduces the interacting Hamiltonian to a diagonalized Hamiltonian.

IX. CONCLUSIONS

In this work we have considered a representation of fermionic and spin operators in terms of Majorana fermions and have used it systematically to relate the two types of problems. We focused on mappings between the two types of systems with particular emphasis on the relation of the topological properties as a result of the unitary transformations that lead from one problem to another. The simpler case of a unitary transformation that relates two fermionic problems was considered and the singular or non-singular nature of the transformation determines the change of topological invariants. The analysis of the change of topological properties due

to the mapping between the interacting spin problem and its fermionic representation is, however, more complex, since the calculation of a topological invariant of the interacting problem in general requires a numerical solution.

The various types of unitary transformations allow mappings to different problems with preserved or change of statistics and in general allow the replacement of interacting terms by free terms and vice-versa. This property has been used in different contexts in the literature to address the problem of strongly interacting systems, and we have considered here the same problem. The application to the XX spin problem is well known and was also expanded on in this work, together with the Hubbard model. The application to the Kondo problem may also be considered but the type of unitary transformations considered here presents some difficulties associated with a spin rotation invariant model. However, the method can be applied to an anisotropic Kondo model and will be considered elsewhere. Another interesting problem may be the effect of interactions in the Kitaev model and its possible mapping to spin problems⁵⁸.

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Appendix A: Relation between states

In this Appendix we summarize relations between the eigenstates of the number operators of the f and g fermions and the action of different operators on them. We begin by considering

$$\begin{aligned} |0\rangle_f &= \frac{1}{\sqrt{2}} (|0\rangle_g + |1\rangle_g) \\ |1\rangle_f &= \frac{i}{\sqrt{2}} (|0\rangle_g - |1\rangle_g) \\ |0\rangle_g &= \frac{1}{\sqrt{2}} (|0\rangle_f - i|1\rangle_f) \\ |1\rangle_g &= \frac{1}{\sqrt{2}} (|0\rangle_f + i|1\rangle_f) \end{aligned} \quad (A1)$$

Therefore,

$$\begin{aligned}
g|0\rangle_f &= \frac{1}{\sqrt{2}}|0\rangle_g \\
g|1\rangle_f &= -\frac{i}{\sqrt{2}}|0\rangle_g \\
g^\dagger|0\rangle_f &= \frac{1}{\sqrt{2}}|1\rangle_g \\
g^\dagger|1\rangle_f &= \frac{i}{\sqrt{2}}|1\rangle_g
\end{aligned} \tag{A2}$$

Also,

$$\begin{aligned}
\gamma_3|0\rangle_f &= -i(f - f^\dagger)|0\rangle_f = i|1\rangle_f \\
\gamma_3|1\rangle_f &= -i|0\rangle_f \\
\gamma_3|0\rangle_g &= \frac{1}{\sqrt{2}}(i|1\rangle_f - |0\rangle_f) = -|0\rangle_g \\
\gamma_3|1\rangle_g &= \frac{1}{\sqrt{2}}(i|1\rangle_f + |0\rangle_f) = |1\rangle_g
\end{aligned} \tag{A3}$$

Finally,

$$\begin{aligned}
g^\dagger g|0\rangle_f &= \frac{1}{\sqrt{2}}|1\rangle_g \\
g^\dagger g|1\rangle_f &= -\frac{i}{\sqrt{2}}|1\rangle_g \\
(2g^\dagger g - 1)|0\rangle_f &= \frac{1}{\sqrt{2}}(-|0\rangle_g + |1\rangle_g) = i|1\rangle_f \\
(2g^\dagger g - 1)|1\rangle_f &= -\frac{i}{\sqrt{2}}(|0\rangle_g + |1\rangle_g) = -i|0\rangle_f
\end{aligned} \tag{A4}$$

Therefore

$$\begin{aligned}
U_z|0\rangle_f &= |0\rangle_f \\
U_z|1\rangle_f &= |1\rangle_f
\end{aligned} \tag{A5}$$

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